VARIABLE VISCOSITY PLANE POISEUILLE FLOW WITH UNEQUAL WALL TEMPERATURES

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(Communicated by F. C. Auluck, F.N.A.)

(*Received* 3 *July* 1973)

A study of the plane Poiseuille flow with unequal wall temperatures of an incompressible fluid having temperature dependent viscosity is made. Using a non-linear viscosity-temperature relation the coupled differential equations of momentum and energy are solved. Expressions for velocity and temperature distributions and for the Nusselt number are obtained.

1. INTRODUCTION

The solutions of the Navier-Stokes equations for the flow between two parallel walls, with constant fluid properties, are well known (Schlichting 1968). However, for the fluids, which are important in the theory of lubrication, the heat generated by the internal friction and the corresponding rise in temperature does affect the viscosity of the fluid and it can no longer be regarded as constant. Nahme (1940) considered the plane Couette flow for a fluid having temperature dependent viscosity and reported that the velocity distribution at right angle to the walls ceases to be linear. Hausenblas (1950), almost after a decade, considered the plane Poiseuille flow taking a simplified form of the viscosity-temperature relation keeping both the walls at the same temperature.

In the present analysis we have reconsidered the problem of Hausenblas (1950) taking the walls at unequal temperatures. One of the important findings of the present investigation is that the maximum velocity does not occur in the middle of the channel, as the case with walls at same temperature or with constant viscosity solutions, but moves towards the upper wall with the rise in the temperature difference between the walls.

The values of the Nusselt number for the transfer of heat at the lower wall are reported in Table I. It is found that the heat transfer in this case is more, in comparison to the constant viscosity flow.

Hausenblas (1950) case and constant viscosity solutions are being easily deduced.

TABLE I

Nusselt number ($T_1 \neq T_2$, $N = 1.0$)

Cases ↓	EP_r →	0.6769	1.4483
Variable viscosity	Nu →	2.2904	2.5028
Constant viscosity	Nu →	1.9328	2.4737

2. Governing Equations and Boundary Conditions

In the case of a steady two-dimensional flow of an incompressible fluid with variable viscosity between two parallel walls, taking axis of x along the central line of the channel and y-axis at right angle to it, the equations governing the motion are (Hausenblas 1950) :

$$\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = \frac{dp}{dx} \qquad \ldots(2.1)$$

$$\kappa \frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 = 0 \qquad \ldots(2.2)$$

and the boundary conditions are

$$\left.\begin{aligned} y = h \quad &: \ u = 0, \quad T = T_1 \\ y = -h \ &: \ u = 0, \quad T = T_0 \\ &T_1 > T_0 \end{aligned}\right\} \qquad \ldots(2.3)$$

where $2h$ is the distance between the walls, and the motion is due to the constant pressure gradient $\frac{dp}{dx}$ along the axis of the channel.

Let us introduce the following non-dimensional quantities :

$$u^* = \frac{u}{u_m}, \ \eta = \frac{y}{h}, \ \mu^* = \frac{\mu}{\mu_0}, \ T^* = \frac{T - T_0}{T_1 - T_0}$$

$P_r = \frac{\mu_0 c_p}{\kappa}$ (Prandtl number) and $E = \frac{u_m^2}{c_p(T_1 - T_0)}$ (Eckert number) where $u_m\left(= -\frac{h^2}{2\mu_0}\frac{dp}{dx}\right)$ is the maximum velocity in the middle of the channel in Plane Poiseuille flow with constant fluid properties (Schlichting 1968), μ_0 is the viscosity of the fluid at temperature T_0.

The eqns. (2.1) and (2.2) take the forms :

$$\frac{d}{d\eta}\left(\mu^* \frac{du^*}{d\eta}\right) = -2 \qquad \ldots(2.4)$$

$$\frac{d^2T^*}{d\eta^2} + \mu^* EP_r \left(\frac{du^*}{d\eta}\right)^2 = 0 \qquad \ldots(2.5)$$

and the corresponding boundary conditions are

$$\left.\begin{array}{l} \eta = 1 \quad : \; u^* = 0, \quad T^* = 1 \\ \eta = -1 \; : \; u^* = 0, \quad T^* = 0. \end{array}\right\} \qquad \ldots(2.6)$$

3. Analysis

Integrating eqn. (2.4), we get

$$\mu^* \frac{du^*}{d\eta} = -\,2(\eta - C), \qquad \ldots(3.1)$$

where C is a constant of integration to be determined.

Let

$$\eta - C = \xi \qquad \ldots(3.2)$$

then eqns. (3.1) and (2.5) may be written as

$$\mu^* \frac{du^*}{d\xi} = -\,2\xi \qquad \ldots(3.3)$$

and

$$\frac{d^2T^*}{d\xi^2} + \mu^* EP_r \left(\frac{du^*}{d\xi}\right)^2 = 0 \qquad \ldots(3.4)$$

respectively.

For the flow of a fluid having temperature dependent viscosity the equations (3.3) and (3.4) are coupled differential equations and in order to solve them we require an empirical relation between viscosity and temperature which is taken to be (Hausenblas 1950)

$$\frac{1}{\mu^*} = 1 + \alpha T^* = \theta^* \text{ (say)} \qquad \ldots(3.5)$$

where

$$\alpha = \frac{b}{T_0^2}(T_1 - T_0) \qquad \ldots(3.6)$$

b being a constant, has the dimension of temperature and depends on the nature of the fluid.

Substituting (3.3) in (3.4) and making use of eqn. (3.5), we get

$$\frac{d^2\theta^*}{d\xi^2} + 4N\xi^2\theta^* = 0 \qquad \ldots(3.7)$$

where

$$N = \alpha E P_r = \frac{u_m^2 \mu_0 b}{\kappa T_0^2} \qquad \text{...(3.8)}$$

is a dimensionless parameter.

The boundary conditions on θ^*, in view of (2.6), (3.2) and (3.5), are :

$$\left.\begin{aligned} \xi &= 1 - C \quad &: \quad \theta^* &= 1 + \alpha, \\ \xi &= -1 - C \quad &: \quad \theta^* &= 1. \end{aligned}\right\} \qquad \text{...(3.9)}$$

The solution of eqn. (3.7), with boundary condition (3.9), is given by

$$\theta^* = 1 + \alpha T^* = A_0 F_1 \{\sqrt{N}\xi^2\} + B_0 \xi F_2 \{\sqrt{N}\xi^2\} \qquad \text{...(3.10)}$$

where

$$F_1(t) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{(-\frac{1}{4})!}{(-\frac{1}{4}+\nu)!} \left(i\frac{t}{2}\right)^{2\nu} \qquad \text{...(3.11)}$$

$$F_2(t) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{(\frac{1}{4})!}{(\frac{1}{4}+\nu)!} \left(i\frac{t}{2}\right)^{2\nu} \qquad \text{...(3.12)}$$

and A_0, B_0 are given by

$$A_0 = \frac{(1+\alpha)(1+C) F_2\{\sqrt{N}(1+C)^2\} + (1-C) F_2\{\sqrt{N}(1-C)^2\}}{(1+C) F_1\{\sqrt{N}(1-C)^2\} F_2\{\sqrt{N}(1+C)^2\} + (1-C) F_1\{\sqrt{N}(1+C)^2\} F_2\{\sqrt{N}(1-C)^2\}} \qquad \text{...(3.13)}$$

$$B_0 = \frac{A_0 F_1\{\sqrt{N}(1+C)^2\} - 1}{(1+C) F_2\{\sqrt{N}(1+C)^2\}}. \qquad \text{...(3.14)}$$

It may be recalled that the constant C is yet to be determined.

Equation (3.3), in view of eqn. (3.5), may be written as

$$\frac{du^*}{d\xi} = -2\xi\theta^*. \qquad \text{...(3.15)}$$

Substituting the value of θ^*, from eqn. (3.10), in eqn. (3.15) and integrating, we find

$$u^* = -2A_0\xi^2 F_3\{\sqrt{N}\xi^2\} - 2B_0\xi^3 F_4\{\sqrt{N}\xi^2\} + C_0 \qquad \text{...(3.16)}$$

where

$$F_3(t) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{(-\frac{1}{4})!}{(-\frac{1}{4}+\nu)!} \frac{1}{(4\nu+2)} \left(i\frac{t}{2}\right)^{2\nu} \qquad \text{...(3.17)}$$

$$F_4(t) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{(\frac{1}{4})!}{(\frac{1}{4}+\nu)!} \frac{1}{(4\nu+3)} \left(i\frac{t}{2}\right)^{2\nu} \qquad \ldots(3.18)$$

and C_0 is a constant of integration to be determined.

The boundary conditions (2.6), in view of (3.2), when applied in (3.16), yield

$$2A_0(1-C)^2 F_3\{\sqrt{N}(1-C)^2\} + 2B_0(1-C)^3 F_4\{\sqrt{N}(1-C)^2\} = C_0 \qquad \ldots(3.19)$$

and

$$2A_0(1+C)^2 F_3\{\sqrt{N}(1+C)^2\} - 2B_0(1+C)^3 F_4\{\sqrt{N}(1+C)^2\} = C_0 \qquad \ldots(3.20)$$

which determine the values of C_0 and C.

Thus the temperature and velocity distributions are given by eqns. (3.10) and (3.16) respectively, where the constants A_0, B_0, C_0 and C are to be obtained from eqns. (3.13), (3.14), (3.19) and (3.20). Eliminating A_0, B_0 and C_0 from these four equations, the equation which will determine C, for prescribed values of α and N, is obtained as follows :

$$\frac{(1+\alpha) F_1\{\sqrt{N}(1+C)^2\} - F_1\{\sqrt{N}(1-C)^2\}}{(1+\alpha)(1+C) F_2\{\sqrt{N}(1+C)^2\} + (1-C) F_2\{\sqrt{N}(1-C)^2\}}$$
$$= \frac{(1+C)^2 F_3\{\sqrt{N}(1+C)^2\} - (1-C)^2 F_3\{\sqrt{N}(1-C)^2\}}{(1+C)^3 F_4\{\sqrt{N}(1+C)^2\} + (1-C)^3 F_4\{\sqrt{N}(1-C)^2\}} \qquad \ldots(3.21)$$

4. Heat Transfer

For the transfer of heat at the lower plate, we define the dimensionless coefficient of heat transfer viz., the Nusselt number Nu as follows :

$$Nu = \frac{-\left(\frac{\partial T}{\partial y}\right)_{y=-h} \cdot h}{(T_0 - T_m)} \qquad \ldots(4.1)$$

where T_m is the temperature in the middle of the channel.

In the present notations eqn. (4.1) may be written as

$$Nu = \frac{\left(\frac{\partial \theta^*}{\partial \xi}\right)_{\xi=-(1+C)}}{(\theta^*_m - 1)} . \qquad \ldots(4.2)$$

Substituting eqn. (3.10) in (4.2), the Nusselt number is obtained as

$$Nu = \frac{-2A_0(1+C)\sqrt{N}F_1'\{\sqrt{N}(1+C)^2\} + B_0[F_2\{\sqrt{N}(1+C)^2\} + 2(1+C)^2\sqrt{N}F_2'\{\sqrt{N}(1+C)^2\}]}{A_0F_1(\sqrt{N}C^2) - B_0CF_2(\sqrt{N}C^2) - 1} \qquad \ldots(4.3)$$

where a prime denotes differentiation with respect to t.

5. Equal Temperature Case

When both the walls are at the same temperature T_0, eqn. (3.6) implies that $\alpha = 0$ but from (3.8) $N \neq 0$. Also due to the symmetry of the flow the value of C will be zero, and therefore from (3.13), (3.14) and (3.19) it follows that

$$A_0 = \frac{1}{F_1(\sqrt{N})}, \quad B_0 = 0 \text{ and } C_0 = \frac{2F_3(\sqrt{N})}{F_1(\sqrt{N})}. \qquad \ldots(5.1)$$

Hence from eqns. (3.10), (3.16) and (4.3) the temperature distribution, the velocity distribution and the Nusselt number reduce to

$$\left.\begin{aligned} \theta^* &= 1 + \frac{b}{T_0^2}(T - T_0) = \frac{F_1(\sqrt{N\eta^2})}{F_1(\sqrt{N})}, \\ u^* &= \frac{2}{F_1(\sqrt{N})}[F_3(\sqrt{N}) - \eta^2 F_3(\sqrt{N\eta^2})] \\ \text{and} \quad Nu &= \frac{2\sqrt{N}F_1'(\sqrt{N})}{F_1(\sqrt{N}) - 1}. \end{aligned}\right\} \qquad \ldots(5.2)$$

The expressions in (5.2) are exactly the same as obtained by Hausenblas (1950).

6. Constant Viscosity Solutions

In the case of constant viscosity $b = 0$, and therefore from eqns. (3.6) and (3.8) we conclude that $\alpha = 0$ and $N = 0$. The equation (3.21) then implies $C = 0$ and from eqns. (3.13), (3.14) and (3.19) it follows that

$$A_0 = 1, \; B_0 = 0 \text{ and } C_0 = 1, \qquad \ldots(6.1)$$

respectively.

Hence eqn. (3.16), for the velocity distribution, reduces to

$$u^* = 1 - \eta^2 \qquad \ldots(6.2)$$

and on evaluating the limits of the expressions in equations (3.10) and (4.3) we find

$$T^* = \frac{T - T_0}{T_1 - T_0} = \tfrac{1}{2}(1 + \eta) + \frac{EP_r}{3}(1 - \eta^4) \qquad \ldots(6.3)$$

and

$$Nu = \frac{3 + 8EP_r}{3 + 2EP_r}. \qquad \ldots(6.4)$$

Further, if the temperatures of both the walls are equal then from (5.2), on evaluating the limit, it follows that

$$T - T_0 = \frac{\mu_0 u_m^2}{3\kappa}(1 - \eta^4) \qquad ...(6.5)$$

and

$$Nu = 4. \qquad ...(6.6)$$

The expressions in equations (6.2) to (6.6) are the same as reported by Pai (1950) and Schlichting (1968).

7. Numerical Discussion

From eqn. (3.21), since C is not explicitly expressible as a function of N and α, it is convenient to determine α for prescribed values of N and C, where $N \geqslant 0$ and $0 \leqslant C \leqslant 1$. The value of α, for two values of C viz. 0.06 and 0.1, have been calculated from eqn. (3.21) when $N = 1.0$. It is found that α is equal to 0.6905 and 1.4774 respectively.

When the walls of the channel are at different temperatures, the velocity and temperature profiles are drawn in Figs. 1 and 2 respectively, and are compared with the profiles of the constant viscosity solution. It is found that in the present case the maximum velocity does not occur in the middle of the channel but moves towards the upper wall as the value of α increases. Moreover, the rise in the magnitude of the velocity is much significant in the present case showing that the volume rate of flow at a section increases with the increase in α. For a given value of EP_r the rise in

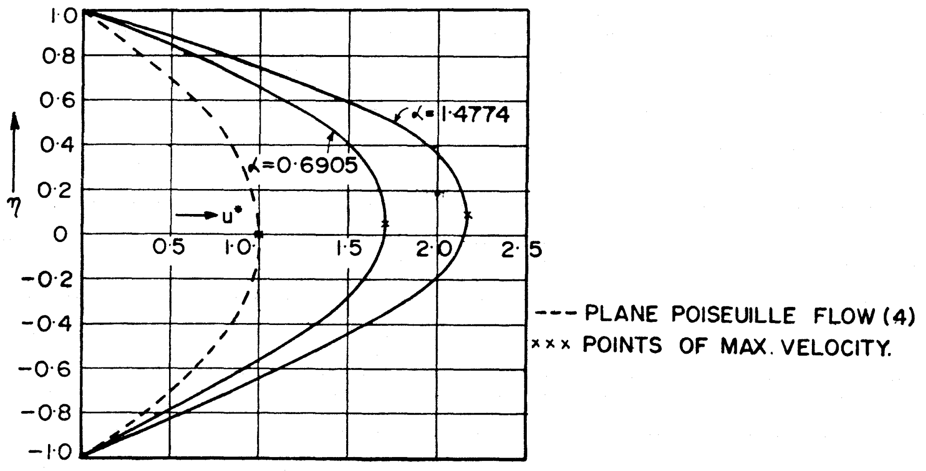

FIG. 1. The velocity distribution, plotted against the perpendicular distance from the central line, for different values of the parameter α ($N = 1.0$, $T_1 \neq T_0$).

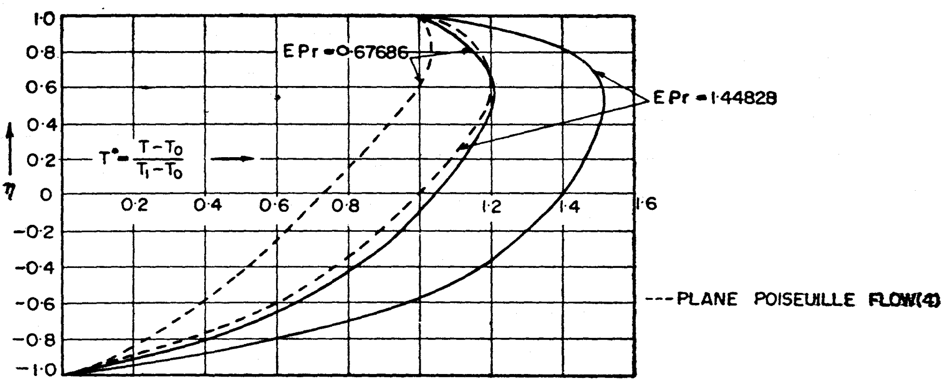

FIG. 2. The temperature distribution, plotted against the perpendicular distance from the central line, for different values of EP_r ($N = 1.0$, $T_1 \neq T_0$).

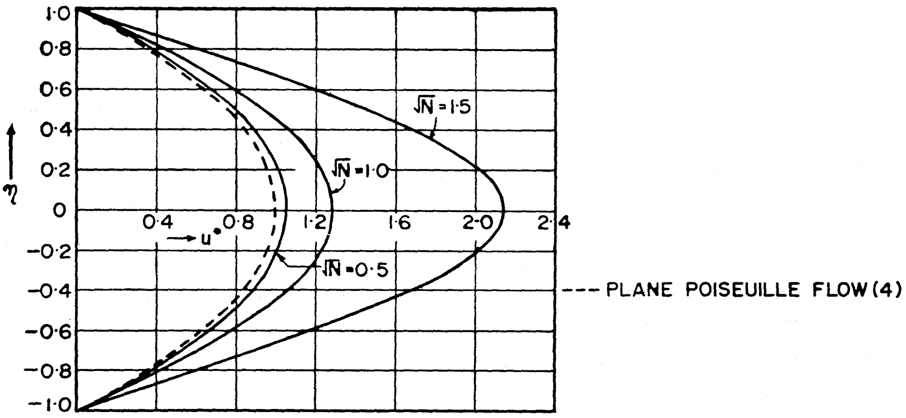

FIG. 3. The velocity distribution, plotted against the perpendicular distance from the central line, for different values of $\sqrt{N}$ ($T_1 = T_0$).

temperature, when compared with the constant viscosity temperature distribution, is also to be noted. This change in the temperature distribution leads to an important conclusion that in the present case the transfer of heat at the lower wall will be more as can also be seen from Table I for the Nusselt number.

Hausenblas (1950) solution, for the case of equal wall temperature, has been deduced in §5 and the corresponding velocity and temperature profiles are drawn in Figs. 3 and 4. For a detailed study of this case reference should be made to his paper.

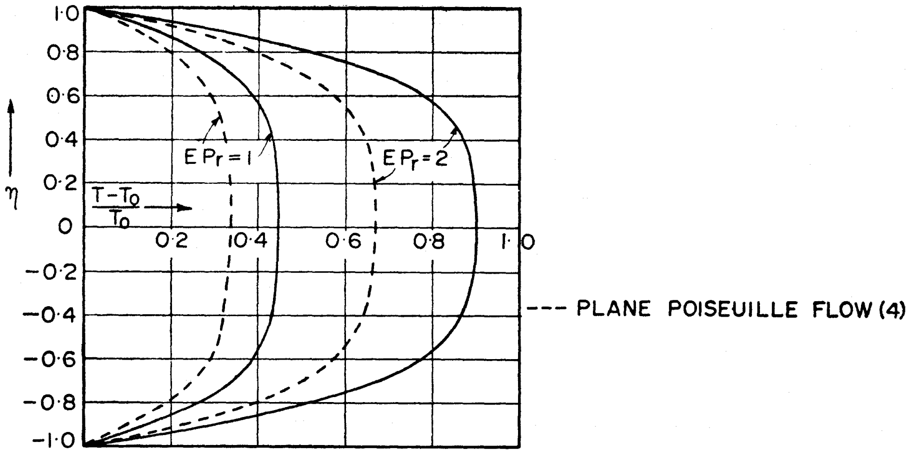

FIG. 4. The temperature distribution, plotted against the perpendicular distance from the central line, for different values of EP_r ($N = 1.0$, $T_1 = T_0$).

ACKNOWLEDGEMENT

The authors are extremely grateful to Professor R. S. Kushwaha for providing research facilities and his kind encouragement. This investigation has been carried out with the financial support of the U.G.C., in the form of a fellowship awarded to one of the authors (N.C.J.).

REFERENCES

Schlichting, H. (1968). Boundary Layer Theory. McGraw-Hill Book Co., Inc., New York.

Nahme, R. (1940). Beiträge zur hydrodynamschen Theorie der Lagerreibung. *Ing.-Arch.*, **11,** 191-209.

Hausenblas, H. (1950). Die nicht isotherme Strömung einer zähen Flüssigkeit durch enge Spalte und Kapillarröhren. *Ing.-Arch.*, **18,** 151.

Pai, S. I. (1950). Viscous Flow Theory—I : Laminar Flow. D. Van Nostrand Co., Amsterdam.