DEFLECTION OF AN ELASTIC CIRCULAR PLATE WITH CLAMPED EDGE RESTING ON A VISCOELASTIC FOUNDATION AND HAVING A CONCENTRATED LOAD AT THE CENTRE

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In the present paper the complete solution of the problem of the deflection of an elastic circular plate with fixed edge resting on a viscoelastic foundation has been obtained with the assumption that there is a concentrated normal load at the centre.

1. INTRODUCTION

Following Reissner (1958) we take the equation of motion of an elastic plate on viscoelastic foundation as

$$D\nabla_1^4 w + K\left(1 + \lambda \frac{\partial}{\partial t}\right) w + \rho h \frac{\partial^2 w}{\partial t^2} = q, \qquad \text{...(1.1)}$$

where

w = transverse displacement of the plate,

D = bending stiffness factor of the plate,

$K\left(1 + \lambda \frac{\partial}{\partial t}\right)$ = modulus of viscoelastic foundation (K and λ being constants),

ρ = uniform density of the plate,

h = its thickness, and

q = transverse load per unit area.

For clamped edge, boundary conditions for a circular plate of radius a are

$$\left.\begin{aligned} &\text{(i) } (w)_{r=a} = 0 \\ &\text{(ii) } \left(\frac{\partial w}{\partial r}\right)_{r=a} = 0. \end{aligned}\right\} \qquad \text{...(1.2)}$$

2. SOLUTION OF THE PROBLEM

To solve eqn. (1.1) we shall first consider the solution of the equation

$$\nabla_1^4 W = k^4 W \text{ (k is a constant).} \qquad \text{...(2.1)}$$

For symmetrical solution, the above equation expressed in polar co-ordinates can be written as

$$\left[\left(\frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr}\right)^2 - k^4\right] W = 0 \qquad \ldots(2.2)$$

The solution of this equation suitable for a complete disc is

$$W = A_1 J_0(kr) + B_1 I_0(kr) \qquad \ldots(2.3)$$

in which J_0 is the Bessel function and I_0 is the modified Bessel function of zero order and of first kind.

The boundary condition (i) of (1.2) gives

$$W = I_0(ka)\, J_0(kr) - J_0(ka)\, I_0(kr) = Z_0(kr) \text{ (say)} \qquad \ldots(2.4)$$

The second boundary condition

$$\left(\frac{\partial w}{\partial r}\right)_{r=a} = 0$$

i.e. $Z_0'(ka) = 0$

or

$$\frac{J_1(ka)}{J_0(ka)} + \frac{I_1(ka)}{I_0(ka)} = 0 \qquad \ldots(2.5)$$

If $k_1, k_2, k_3, \ldots, k_n, \ldots$, are the roots of this equation, then $W = Z_0(k_1 r), Z_0(k_2 r), Z_0(k_3 r), \ldots, Z_0(k_n r), \ldots$ are solutions of the equation (2.1) which satisfy the boundary conditions.

Let us put

$$w = \sum_{n=1}^{\infty} A_n(t)\; Z_0(k_n r) \qquad \ldots(2.6)$$

Again if $q = P\delta(r)/2\pi r$ where P is a constant, and $\delta(r)$ is Dirac delta function, we can write

$$P\delta(r)/2\pi r = \sum_{n=1}^{\infty} B_n Z_0(k_n r) \qquad \ldots(2.7)$$

B_n being, a constant.

It can be readily shown that the function $Z_0(k_n r)$ are orthogonal with the following properties similar to those for Bessel functions :

$$\int_0^a rZ_0(k_nr)\, Z_0(k_mr)\, dr \simeq 0$$

$$\int_0^a rZ_0^2\,(k_nr)\, dr = \tfrac{1}{2}\, a^2\, [I_0^2\,(k_na)\, J_1^2\,(k_na) - J_0^2\,(k_na) I_1^2\,(k_na)$$
$$+ 2\, J_0^2\,(k_na)\, I_0^2\,(k_na)]. \qquad \ldots(2.8)$$

Multiplying both sides of (2.7) by $rZ_0(k_nr)\, dr$, we obtain from (2.7),

$$\frac{P}{2\pi} \int_0^a \delta(r)\, Z_0(k_nr)\, dr = B_n \int_0^a rZ_0^2\,(k_nr)\, dr$$

or

$$B_n = \frac{P}{2\pi} \cdot \frac{[I_0(k_na) - J_0(k_na)]}{\frac{a^2}{2}\, [I_0^2\,(k_na)\, J_1^2\,(k_na) - J_0^2\,(k_na)\, I_1^2\,(k_na) + 2\, J_0^2\,(k_na)\, I_0^2\,(k_na)]}. \qquad \ldots(2.9)$$

Putting

$$w = \sum_{n=1}^{\infty} A_n(t)\, Z_0(k_nr) \text{ and writing } q = \sum_{n=1}^{\infty} B_nZ_0(k_nr)$$

we obtain eqn. (1.1) as

$$D \sum_{n=1}^{\infty} A_n(t)\, k_n^4\, Z_0(k_nr) + K\left(1 + \lambda\, \frac{\partial}{\partial t}\right) \sum_{n=1}^{\infty} A_n(t)\, Z_0(k_nr)$$
$$+ \rho h \sum_{n=1}^{\infty} \frac{d^2A_n(t)}{dt^2}\, Z_0(k_nr) = \sum_{n=1}^{\infty} B_nZ_0(k_nr).$$

This will be satisfied if for all values of $\eta = 1, 2, 3, \ldots,$

$$DA_n(t)\, k_n^4 + K\left(1 + \lambda\, \frac{\partial}{\partial t}\right) A_n(t) + \rho h\, \frac{d^2A_n(t)}{dt^2} = B_n.$$

Taking Laplace transform and using the symbol

$$\bar{\varphi}\,(p) = \int_0^{\infty} e^{-pt}\, \varphi(t)\, dt$$

we have

$$[(Dk_n^4 + K) + K\lambda p + \rho hp^2]\, \bar{A}_n(p) = \frac{B_n}{p}\, .$$

It is assumed that $w = \frac{\partial w}{\partial t} = 0$, when $t = 0$.

$$\therefore \bar{A}_n(p) = \frac{B_n}{p\,[\rho h p^2 + K\lambda p + (Dk_n^4 + K)]}$$

$$= B_n \left[\frac{1}{(Dk_n^4 + K)\,p} - \frac{p\rho h}{(Dk_n^4 + K)\,\{\rho h p^2 + K\lambda p + (Dk_n^4 + K)\}} - \frac{K\lambda}{(Dk_n^4 + K)\,\{\rho h p^2 + K\lambda p + (Dk_n^4 + K)\}} \right]$$

$$= B_n \left[\frac{1}{(Dk_n^4 + K)p} - \frac{1}{Dk_n^4 + K} \cdot \frac{p + \frac{K\lambda}{2\rho h}}{\left(p + \frac{K\lambda}{2\rho h}\right)^2 + \left(\frac{Dk_n^4 + K}{\rho h} - \frac{K^2\lambda^2}{4\rho^2 h^2}\right)} - \frac{1}{Dk_n^4 + K} \cdot \frac{K\lambda}{2\rho h} \cdot \frac{1}{\left(p + \frac{K\lambda}{2\rho h}\right)^2 + \left(\frac{Dk_n^4 + K}{\rho h} - \frac{K^2\lambda^2}{4\rho^2 h^2}\right)} \right].$$

Taking inverse transform we get

$$A_n(t) = B_n \left[\frac{1}{Dk_n^4 + K} - \frac{1}{Dk_n^4 + K}\, e^{-(K\lambda/2\rho h)t} \cos \sqrt{\frac{Dk_n^4 + K}{\rho h} - \frac{K^2\lambda^2}{4\rho^2 h^2}}\; t - \frac{1}{Dk_n^4 + K} \cdot \frac{K\lambda}{2\rho h}\, e^{-(K\lambda/2\rho h)t} \sin \sqrt{\frac{Dk_n^4 + K}{\rho h} - \frac{K^2\lambda^2}{4\rho^2 h^2}}\; t \right]$$

$$\therefore w = \sum_{n=1}^{\infty} B_n Z_0(k_n r) \left[\frac{1}{Dk_n^4 + K} - \frac{1}{Dk_n^4 + K}\, e^{-(K\lambda/2\rho h)t} \times \cos \sqrt{\frac{Dk_n^4 + K}{\rho h} - \frac{K^2\lambda^2}{4\rho^2 h^2}}\; t - \frac{K\lambda}{\rho h\left(Dk_n^4 + K\right)} \times e^{-(K\lambda/2\rho h)t} \sin \sqrt{\frac{Dk_n^4 + K}{\rho h} - \frac{K^2\lambda^2}{4\rho^2 h^2}}\; t \right].$$

Roots of eqn. (2.5) are

$$k_1 a = 3.196,\ k_2 a = 6.306,\ k_3 a = 9.440,$$
$$k_4 a = 12.577,\ k_5 a = 15.716, \text{ etc.} \qquad \ldots(2.10)$$

For large t i.e. when $t \to \infty$

$$w = \sum_{n=1}^{\infty} B_n Z_0(k_n r) \frac{1}{Dk_n^4 + K}$$

$$= \frac{P}{2\pi} \sum_{n=1}^{\infty} \frac{I_0(k_n a) - J_0(k_n a)}{\frac{a^2}{2}\left[I_0^2(k_n a)\, J_1^2(k_n a) - J_0^2(k_n a)\, I_1^2(k_n a) + 2\, J_0^2(k_n a)\, I_0^2(k_n a)\right]}$$

$$\times \frac{Z_0(k_n r)}{Dk_n^4 + K}. \qquad \ldots(2.11)$$

Since

$$[Z_0(k_n r)]_{r=0} = I_0(ka) - J_0(ka) \quad [\text{vide } (2.4)]$$

we have from (2.10)

$$w = \frac{P}{\pi} \cdot \frac{5{\cdot}4\, a^4}{a^2(D\, 104{\cdot}8 + Ka^4)} \text{ (approx.).}$$

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