ON THE DEGREE OF APPROXIMATION TO A FUNCTION BY TRIANGULAR MATRIX OF ITS FOURIER SERIES

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Siddiqi (1971) proved a result on the degree of approximation to a function by the Cesàro means of its Fourier series. In the present study Siddiqi's result has been extended in terms of the more general triangular summability of which (C, α), $\alpha > 0$ is a special case.

§1. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let its Fourier series be given by

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

We shall use the following notations :

$$\phi(t) = \phi_x(t) = f(x+t) + f(x-t) - 2f(x)$$

$$\Phi(t) = \int_0^t | \phi(u) | \, du.$$

Let $(\lambda_{n,k})$ $(n = 0, 1, \ldots, \; k = 0, 1, \ldots n, \; \lambda_{n,0} = 1)$ be a triangular matrix of real or complex numbers.

Let

$$\sigma_n = \sum_{k=0}^{n} \lambda_{n,k} U_k$$

$$= \sum_{k=0}^{n} \Delta \lambda_{n,k} S_k \qquad \ldots(1.1)$$

where

$$\Delta \lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}$$

and

$$\Delta^2 \lambda_{n,k} = \Delta \lambda_{n,k} - \Delta \lambda_{n,k+1}.$$

A series ΣU_n with partial sum s_n is said to be summable $(\wedge)$ to s if the sequence $\{\sigma_n\}$ tends to a finite limit s as n tends to infinity.

The necessary and sufficient conditions for the $(\wedge)$ to be regular are that

(a) there is a constant M such that

$$\sum_{k=0}^{\infty} | \triangle \lambda_{n,k} | < M \text{ for every } n,$$

(b) for every k,

$$\lim_{n\to\infty} \triangle \lambda_{n,k} = 0,$$

(c) $\lim_{n\to\infty} \sum_{k=0}^{\infty} \triangle \lambda_{n,k} = 1.$

In particular if

$$\triangle \lambda_{n,k} = \frac{p_{n-k}}{P_n} \quad (k \leqslant n)$$
$$= 0 \qquad (k > n)$$

where $P_n = \sum_{\nu=0}^{n} p_\nu \neq 0$ then σ_n defined by (1.1) is the same as Nörlund mean generated by the sequence of coefficients $\{p_n\}$ usually denoted as (N, p_n) mean.

Similarly if

$$\triangle \lambda_{n,k} = \frac{\binom{n-k+\alpha-1}{\alpha-1}}{\binom{n+\alpha}{\alpha}}, \ \alpha > 0, \text{ for } k \leqslant n$$
$$= 0 \qquad \text{for } k > n,$$

then σ_n mean is the same as the (C, α) mean, the familiar Cesàro means of order $\alpha > 0$.

§2. Siddiqi (1971) proved the following theorem on the degree of approximation to a function by Cesàro means of its Fourier series.

Theorem A — Let $0 < k < 1$ and $0 < \delta \leqslant \pi$. If x is a point such that $\int_0^t | d\phi(u) | \leqslant A\, \psi(t)$ where $0 \leqslant t \leqslant \delta$, then $\sigma_n^k(x) - f(x) = O\left(\psi\left(\frac{1}{n}\right)\right) + O(n^{-k})$

where $\sigma_n^k(x)$ is the Cesàro means of order k, and $\psi(t)$ is a positive increasing function such that

$$\int_{1/n}^{\delta} \frac{\psi(t)}{t^2}\, dt = O\left(n\, \psi\left(\frac{1}{n}\right)\right), \; n \to \infty.$$

The object of this paper is to extend the result of Siddiqi (1971) in terms of the more general triangular summability of which (C, α) is a special case. We prove as follows :

Theorem — If

$$\int_{0}^{t} |\ d\phi(u)\ | \leqslant A\psi(t) \text{ where } 0 \leqslant t \leqslant \delta \qquad \text{...(2.1)}$$

then

$$\sigma_n(x) - f(x) = O\left(\psi\left(\frac{1}{n}\right)\right) \qquad \text{...(2.2)}$$

where $\psi(t)$ is a positive increasing function such that

$$\int_{1/n}^{\delta} \frac{\psi(t)}{t^2}\, dt = O\left(n\, \psi\left(\frac{1}{n}\right)\right), \; n \to \infty \qquad \text{...(2.3)}$$

In order to prove the theorem we will use the following lemmas.

Lemma 1 — If $\{\triangle\, \lambda_{n,k}\}_{k=0}^{n}$ is non-negative and non-decreasing with respect to k, then

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \triangle^2\, \lambda_{n,k} = 0.$$

The proof is due to the condition of regularity.

Lemma 2 (Nand Kishore 1971) — If $\{\triangle\, \lambda_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and for every n

$$\left| \sum_{k=a}^{b} \triangle\, \lambda_{n,n-k}\, e^{i(n-k)t} \right| < Bt^{-1}\, \triangle\, \lambda_{n,n-\tau}$$

where τ is integral part of $1/t$.

Lemma 3 (Nand Kishore 1971) — If $\{\triangle\, \lambda_{n,k}\}_{k=0}^{n}$ is non-negative and non-decreasing sequence with respect to k such that $\sum_{k=0}^{n} \triangle\, \lambda_{n,k} = 1$ then as $n \to \infty$

$$\triangle \lambda_{n,k} = O\left(\frac{1}{n-k+1}\right)$$

uniformly for all $k \leqslant n$ so that

$$\triangle \lambda_{n,0} = O\left(\frac{1}{n}\right).$$

§3. *Proof of the Theorem* — Let us write

$$S_k(x) = \tfrac{1}{2} a_0 + \sum_{\nu=1}^{k} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

$$S_k(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin\left(\frac{2k+1}{2}\right)t}{\sin t/2} dt$$

then we have

$$\sigma_n(x) - f(x) = \sum_{k=0}^{n} \triangle \lambda_{n,k} \{S_k(x) - f(x)\}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{n} \triangle \lambda_{n,k} \int_0^\pi \phi(t) \frac{\sin\left(\frac{2k+1}{2}\right)t}{\sin t/2} dt$$

$$= \int_0^\pi \phi(t) \frac{1}{2\pi} \sum_{k=0}^{n} \triangle \lambda_{n,k} \frac{\sin\left(\frac{2k+1}{2}\right)t}{\sin t/2} dt$$

$$= \int_0^\pi \phi(t) K_n(t) dt$$

where

$$K_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} \triangle \lambda_{n,k} \frac{\sin\left(\frac{2k+1}{2}\right)t}{\sin t/2}.$$

Now

$$\int_0^\pi \phi(t) K_n(t) dt = \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi \right\} \phi(t) K_n(t) dt$$

$$= I_1 + I_2 + I_3 \text{ (say).}$$

(i) Now uniformly in $0 < t \leqslant 1/n$

$$| K_n(t) | = \frac{1}{2\pi} \left| \sum_{k=0}^{n} \triangle \lambda_{n,k} \frac{\sin \left(\frac{2k+1}{2}\right) t}{\sin t/2} \right|$$

$$\leqslant \frac{1}{2\pi} \sum_{k=0}^{n} | \triangle \lambda_{n,k} | (2k+1).$$

Applying Abel's lemma we get

$$| K_n(t) | \leqslant \frac{1}{2\pi} \left\{ \sum_{k=0}^{n-1} \left(\sum_{0}^{k} | \triangle \lambda_{n,k} | \right) | 2k+1-2k-3 | \right.$$

$$\left. + (2n+1) \sum_{0}^{n} | \triangle \lambda_{n,k} | \right\}$$

$$\leqslant \frac{1}{2\pi} \{2\, Mn + (2n+1)\, M\}.$$

Hence $\quad K_n(t) = O(n)$ as $n \to \infty$.

Since $\quad \phi(0) = 0$, we get

$$| \phi(t) | \leqslant \int_0^t | d\, \phi(u) | \leqslant A \psi(t) \text{ by hypothesis.}$$

Therefore

$$I_1 = O\left(n \int_0^{1/n} \psi(t)\, dt \right)$$

$$= O\left(\psi\left(\frac{1}{n}\right)\right).$$

(ii) Now

$$I_2 = \int_{1/n}^{\delta} \phi(t)\, K_n(t)\, dt$$

$$= \left| \int_{1/n}^{\delta} \phi(t) \frac{1}{2\pi} \sum_{k=0}^{n} \triangle \lambda_{n,k} \frac{\sin \left(\frac{2k+1}{2}\right) t}{\sin t/2}\, dt \right| \leqslant$$

(*equation continued on p.* 854)

$$\leqslant \int_{1/n}^{\delta} | \phi(t) | \frac{1}{2\pi} \left| \sum_{k=0}^{n} \triangle \lambda_{n,n-k} \frac{\sin\left(\frac{2n-2k+1}{2}\right)t}{\sin t/2} dt \right|$$

$$= \frac{1}{2\pi} \int_{1/n}^{\delta} | \phi(t) | \left| \sum_{k=0}^{n} \triangle \lambda_{n,n-k} \frac{\sin\left(\frac{2n-2k+1}{2}\right)t}{\sin t/2} dt \right|$$

$$\leqslant \frac{1}{2\pi} \int_{1/n}^{\delta} | \phi(t) | Bt^{-1} \triangle \lambda_{n,n-r} \frac{1}{t} dt \text{ (by Lemma 2)}$$

$$= \frac{1}{2\pi} \int_{1/n}^{\delta} \frac{\psi(t)}{t^2} B \triangle \lambda_{n,n-r} dt$$

$$\leqslant \frac{B}{2\pi} \int_{1/n}^{\delta} \frac{\psi(t)}{t^2} \cdot \triangle \lambda_{n,0} dt$$

$$= O\left(n\psi\left(\frac{1}{n}\right)\right) \cdot \left(\frac{1}{n}\right) \text{ (by Lemma 3 and condition 2.3)}$$

$$= O\left(\psi\left(\frac{1}{n}\right)\right).$$

(iii) Now

$$| K_n(t) | = \frac{1}{2\pi} \left| \sum_{k=0}^{n} \triangle \lambda_{n,k} \frac{\sin\left(\frac{2k+1}{2}\right)t}{\sin t/2} \right|$$

$$\leqslant \frac{1}{2\pi} \sum_{k=0}^{n-1} | \triangle^2 \lambda_{n,k} | \left| \frac{\sin^2\left(\frac{2k+1}{2}\right)t}{\sin^2 t/2} \right|$$

$$+ \frac{1}{\pi} | \triangle \lambda_{n,n} | \left| \frac{\sin^2\left(\frac{2n+1}{2}\right)t}{\sin^2 t/2} \right|$$

$$\max_{0\leqslant\delta\leqslant t\leqslant\pi} | K_n(t) | < \frac{1}{2\pi \sin^2 \delta/2} \left[\sum_{k=0}^{n-1} | \triangle \lambda_{n,k} | + o(1) \right]$$

$$= o(1) \text{ as } n \to \infty.$$

$$I_3 \leqslant \int_\delta^\pi \psi(t) \mid K_n(t) \mid dt$$

$$< \max_{0\leqslant\delta\leqslant t\leqslant\pi} | K_n(t) | \int_\delta^\pi \psi(t)\, dt$$

$$= o(1) \text{ as } n \to \infty.$$

Therefore

$$\sigma_n(x) - f(x) = O\left(\psi \left(\frac{1}{n} \right)\right)$$

which completes the proof of the theorem.

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