BASES IN A CERTAIN SPACE OF DIRICHLET ENTIRE TRANSFORMATIONS

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The authors consider here the family χ of all entire functions f represented by Dirichlet series and having finite growth (ρ, A). The characterization of such functions in terms of their Dirichlet coefficients makes it possible to introduce in a natural way a locally convex topology $\mathcal{T}$ on χ such that χ becomes a Fréchet space. The main aim in this paper is to characterize proper bases in $(\chi, \mathcal{T})$ in terms of their growth conditions.

1. INTRODUCTION AND TERMINOLOGY

Let χ denote the space of all entire functions defined by Dirichlet series, equipped with the topology of uniform convergence in closed vertical strips. When $f \in \chi$, it means (and conversely as well) that $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ is the complex plane equipped with the usual topology, such that

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \; s = \sigma + it \in \mathbb{C}, \qquad \ldots(1.1)$$

where $0 < \lambda_1 < \ldots < \lambda_n < \ldots \to \infty$ with n, and further

$$\limsup_{n\to\infty} \frac{\log |\, a_n \,|}{\lambda_n} = -\infty \qquad \ldots(1.2)$$

$$\lim_{n\to\infty} \frac{\log \lambda_n}{n} = D < +\infty. \qquad \ldots(1.3)$$

Earlier we have discussed (Kamthan and Gautam 1972) a characterization of proper bases $\{\alpha_n\}$ in χ in terms of the growth properties of maximum modulii of the functions $\alpha_n \in \chi$. Our aim in this note is similar. But this time we single out a subclass of χ and then discuss the characterization of proper bases $\{\alpha_n\}$ in terms of the growth of the norms on α_n.

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To state our results very precisely let ρ and A be two non-zero finite real numbers and χ denote the family of all entire Dirichlet functions f, such that for each $f \in \chi$

$$\limsup_{\sigma \to \infty} \frac{\log M(\sigma, f)}{e^{\rho\sigma}} \leqslant A < +\infty \qquad \ldots(1.4)$$

where

$$M(\sigma, f) = \sup_{-\infty < t < +\infty} | f(\sigma + it) | .$$

Functions of this class have order (R) at the most equal to ρ and type, with respect to ρ, at the most equal to A. It is not difficult to check that functions in χ can be characterized by the following condition, namely, an $f \in \chi$ if and only if

$$\limsup_{n \to \infty} \lambda_n^{1/\rho} | a_n |^{1/\lambda_n} \leqslant (Ae\rho)^{1/\rho} \qquad \ldots(1.5)$$

[see for instance Theorem 1 of Kamthan (1962) where we put $\rho(\sigma) = \rho$ and $\phi(t) = t^{1/\rho}$].

For the sake of simplicity, write $\phi(\lambda_n)$ for $\lambda_n^{1/\rho}$. Define then for each $f \in \chi$, the quantity $\| f; A + \delta \|$, where $\delta > 0$ and

$$\| f; A + \delta \| = \sum_{n=1}^{\infty} | a_n | \left\{ \frac{\phi(\lambda_n)}{((A + \delta)\, e\rho)^{1/\rho}} \right\}^{\lambda_n} . \qquad \ldots(1.6)$$

Clearly (1 6) exists in view of (1.5) for each $\delta > 0$. We denote by $\chi(A, \rho, \delta)$ the space χ equipped with the norm $\| \ldots, A + \delta \|$. The family $\{\| f; A + \delta \| : \delta > 0\}$ generates a locally convex topology $\mathcal{T}$ on χ such that χ becomes a Fréchet space (i.e., a complete metrizable locally convex topological vector space), the equivalent metric λ on χ being given by

$$\lambda(f, g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\| f - g; A + 1/q \|}{1 + \| f - g; A + 1/q \|}$$

[we leave the details of these facts, for they are easily obtained on following the techniques of Theorem 1 of Husain and Kamthan (1968)]. Throughout the rest of this paper we assume that χ is equipped by the topology $\mathcal{T}$, or, equivalently by the topology generated by the invariant metric λ.

A sequence $\{\alpha_n\} \subset \chi$ will be called linearly independent if $\sum_{n=1}^{\infty} a_n \alpha_n = 0$ implies $a_n = 0$ for all sequences $\{a_n\}$ of complex numbers for which $\sum_{n=1}^{\infty} a_n \alpha_n$ converges in χ. The sequence $\{\alpha_n\}$ 'spans' a subspace χ_0 of χ, if χ_0 consists of all linear combinations

$\sum_{n=1}^{\infty} a_n\alpha_n$, such that $\sum_{n=1}^{\infty} a_n\alpha_n$ converges in χ. A sequence $\{\alpha_n\}$ which is linearly independent and spans a closed subspace χ_0 of χ, will be said to be a base in χ_0. If $e_n \in \chi$, $e_n(s) = e^{s\lambda_n}$, $n \geqslant 1$, then clearly $\{e_n\}$ is a base in χ. A sequence $\{\alpha_n\} \subset \chi$ will be called a 'proper base' if it is a base and it satisfies the condition, "for all sequences $\{a_n\}$ of complex numbers, convergence of $\sum_{n=1}^{\infty} a_n\alpha_n$ in χ implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in χ".

We know that $\sum_{n=1}^{\infty} a_n e_n$ converges in χ is equivalent to the condition (see Husain and Kamthan 1968, p. 210).

$$\limsup_{n\to\infty} |\, a_n \,|^{1/\lambda_n} \phi(\lambda_n) \leqslant (Ae\rho)^{1/\rho}. \qquad \ldots(1.7)$$

2. Theorem

Theorem 1 — A necessary and sufficient condition that there exists a continuous linear transformation $T : \chi \to \chi$ with $T(e_n) = \alpha_n$, $n = 1, 2, \ldots$, where $\alpha_n \in \chi$, is that for each $\delta > 0$,

$$\limsup_{n\to\infty} \frac{\|\alpha_n; A + \delta\|^{1/\lambda_n}}{\phi(\lambda_n)} < \frac{1}{(Ae\rho)^{1/\rho}}. \qquad \ldots(2.1)$$

Proof : Let T be a continuous linear transformation from χ into χ with $T(e_n) = \alpha_n$, $n = 1, 2, \ldots$. Then for each $\delta > 0$, there exists a $\delta_1 > 0$, and a constant $K = K(\delta)$, depending on δ, such that

$$\| T(e_n); A + \delta \| \leqslant K \| e_n; A + \delta_1 \|$$

$$\Rightarrow \| \alpha_n; A + \delta \| \leqslant K \left\{ \frac{\phi(\lambda_n)}{((A + \delta_1)\, e\rho)^{1/\rho}} \right\}^{\lambda_n}$$

$$\Rightarrow \frac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} \leqslant \frac{1 + o(1)}{((A + \delta_1)\, e\rho)^{1/\rho}}, \text{ for all } n \geqslant N$$

$$\Rightarrow \limsup_{n\to\infty} \frac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} < \frac{1}{(Ae\rho)^{1/\rho}}.$$

Conversely, let $\{\alpha_n\}$ satisfy (2.1) and let $\alpha = \sum_{n=1}^{\infty} a_n e_n$. This implies that

$$\limsup_{n\to\infty} \phi(\lambda_n) \,|\, a_n \,|^{1/\lambda_n} \leqslant (Ae\rho)^{1/\rho}.$$

So given $\eta > 0$, there exists $N_0 = N_0(\eta)$, such that

$$\phi(\lambda_n) \mid a_n \mid^{1/\lambda_n} \leqslant ((A + \eta)e\rho)^{1/\rho}, \text{ for all } n \geqslant N_0$$

and taking $\eta' > \eta$, we can find, $N_1 = N_1(\eta')$, from (2.1), such that, for $n \geqslant N_1$

$$\frac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} \leqslant \frac{1}{((A + \eta')\, e\rho)^{1/\rho}}.$$

Now choose $n \geqslant \max(N_0, N_1)$, then

$$\mid a_n \mid \cdot \| \alpha_n; A + \delta \| \leqslant \left\{\frac{(A + \eta)\, e\rho}{(A + \eta')\, e\rho}\right\}^{\lambda_n/\rho}.$$

Since $\eta' > \eta$, the series $\sum_{n=1}^{\infty} \mid a_n \mid \cdot \| \alpha_n; A + \delta \|$ converges for each $\delta > 0$. Therefore $\sum_{n=1}^{\infty} a_n\alpha_n$ converges to an element of χ. Define $T(\alpha) = \sum_{n=1}^{\infty} a_n\alpha_n$, for each $\alpha \in \chi$, then $T(e_n) = \alpha_n$. We have only to prove the continuity of T. Given $\delta > 0$, there exists $\delta' > 0$, such that

$$\frac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} \leqslant \frac{1}{((A + \delta')\, e\rho)^{1/\rho}}, \text{ for all } n \geqslant N = N(\delta, \delta').$$

and so

$$\| \alpha_n; A + \delta \| \leqslant K \left\{\frac{\phi(\lambda_n)}{((A + \delta')\, e\rho)^{1/\rho}}\right\}^{\lambda_n}, \text{ with } K = K(\delta)$$

and the inequality is true for all $n \geqslant 0$. Now

$$\begin{aligned} \| T(\alpha); A + \delta \| &\leqslant \sum_{n=1}^{\infty} \mid a_n \mid \cdot \| \alpha_n; A + \delta \| \\ &\leqslant K \sum_{n=1}^{\infty} \mid a_n \mid \left\{\frac{\phi(\lambda_n)}{((A + \delta')\, e\rho)^{1/\rho}}\right\}^{\lambda_n} \\ &= K \| \alpha_n; A + \delta' \|. \end{aligned}$$

Hence T is continuous.

Proper Bases and their Characterizations

Lemma 1 — The following three properties are equivalent :

(A) $\limsup_{n\to\infty} \dfrac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} < \dfrac{1}{(Ae\rho)^{1/\rho}}$, for each $\delta > 0$.

(B) For all sequences $\{a_n\}$ of complex numbers, $\sum_{n=1}^{\infty} a_n e_n$ converges in χ implies $\sum_{n=1}^{\infty} a_n\alpha_n$ converges in χ.

(C) For all sequences $\{a_n\}$ of complex numbers, $\sum_{n=1}^{\infty} a_n e_n$ converges in χ implies $a_n \alpha_n$ tends to zero on χ.

PROOF : It is clear that $(B) \Rightarrow (C)$. We have already proved in the course of the proof of sufficiency part of Theorem 1, that $(A) \Rightarrow (B)$. We have therefore only to prove that (C) implies (A). Assume that (C) is true and that (A) is false. This implies that for a particular δ, say δ', there exists a sequence $\{n_k\}$ of positive integers, such that

$$\frac{\| \alpha_{n_k} ; A + \delta' \|^{1/\lambda_{n_k}}}{\phi(\lambda_{n_k})} \geqslant \frac{1}{\left(\left(A + \frac{1}{k}\right) e\rho\right)^{1/\rho}}, \text{ for all } n_k, k = 1, 2, \ldots \ldots (2.2)$$

Define $\{a_n\}$ by,

$$a_n = \begin{cases} \dfrac{1}{\| \alpha_n ; A + \delta' \|}, & \text{when } n = n_k; \\ 0, & \text{when } n \neq n_k. \end{cases} \quad \ldots (2.3)$$

So we have, for large k

$$| a_{n_k} |^{1/\lambda_{n_k}} \phi(\lambda_{n_k}) = \frac{\phi(\lambda_{n_k})}{\| \alpha_{n_k} ; A + \delta' \|^{1/\lambda_{n_k}}}.$$

$$\leqslant \left(\left(A + \frac{1}{k}\right) e\rho\right)^{1/\rho}, \text{ for all } k \geqslant k_0.$$

Hence,

$$\limsup_{k \to \infty} | a_{n_k} |^{1/\lambda_{n_k}} \phi(\lambda_{n_k}) \leqslant (Ae\rho)^{1/\rho}.$$

Thus $\{a_n\}$ defined by (2.3) satisfy (1.7), but on the contrary,

$$\| a_{n_k} \alpha_{n_k} ; A + \delta' \| = | a_{n_k} | \cdot \| \alpha_{n_k} ; A + \delta' \| = 1$$

So $a_n \alpha_n$ does not tend to zero in $\chi(\rho, A, \delta')$, a contradiction, and this proves that $(C) \Rightarrow (A)$.

Lemma 2 — The following three properties are equivalent :

(α) $$\lim_{\delta \to 0} \left\{ \liminf_{n \to \infty} \frac{\| \alpha_n ; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} \right\} \geqslant \frac{1}{(Ae\rho)^{1/\rho}}.$$

(β) For all sequences $\{a_n\}$ of complex numbers $\sum_{n=1}^{\infty} a_n \alpha_n$ converges in χ implies that $\sum_{n=1}^{\infty} a_n e_n$ converges in χ.

(γ) For all sequences $\{a_n\}$ of complex numbers $a_n\alpha_n$ tends to zero in χ implies that $\sum_{n=1}^{\infty} a_n e_n$ converges in χ.

PROOF: It is clear that (γ) implies (β). We shall prove that (β) implies (α) and (α) implies (γ).

First, we suppose that (β) holds but (α) does not hold. Therefore

$$\lim_{\delta\to 0}\left\{\liminf_{n\to\infty}\frac{\|\alpha_n; A+\delta\|^{1/\lambda_n}}{\phi(\lambda_n)}\right\} < \frac{1}{(Ae\rho)^{1/\rho}}. \qquad \ldots(2.4)$$

Since, $\|\alpha_n; A+\delta\|$ increases as δ decreases, this implies that for each $\delta > 0$,

$$\liminf_{n\to\infty}\frac{\|\alpha_n; A+\delta\|^{1/\lambda_n}}{\phi(\lambda_n)} < \frac{1}{(Ae\rho)^{1/\rho}}. \qquad \ldots(2.5)$$

Hence, if η be a fixed small positive number, for each $r > 0$, we can find a positive number n_r, such that, for all r, we have $n_{r+1} > n_r$ and,

$$\frac{\|\alpha_{n_r}; A+(1/r)\|^{1/\lambda_{n_r}}}{\phi(\lambda_{n_r})} \leqslant \frac{1}{((A+\eta)\,e\rho)^{1/\rho}}. \qquad \ldots(2.6)$$

Now choose a positive number $\eta_1 < \eta$, and define a sequence $\{a_n\}$ by

$$a_n = \begin{cases} \left\{\dfrac{((A+\eta_1)\,e\rho)^{1/\rho}}{\phi(\lambda_n)}\right\}^{\lambda_n}, & \text{when } n = n_r,\ r > r_0 \\ 0, & \text{when } n \neq n_r,\ n < n_{r_0}. \end{cases}$$

Then for any $\delta > 0$,

$$\sum_{n=1}^{\infty} |a_n| \cdot \|\alpha_n; A+\delta\| = \sum_{r=1}^{\infty} |a_{n_r}| \cdot \|\alpha_{n_r}; A+\delta\|. \qquad \ldots(2.7)$$

Given $\delta > 0$, omit from the above series those finite number of terms, which correspond to those number n_r, for which $1/r$ is greater than δ. The remainder of the series is dominated by

$$\sum_{r=1}^{\infty} |a_{n_r}| \cdot \|\alpha_{n_r}; A+(1/r)\|.$$

Now, by (2.6) and (2.7), we see that

$$\sum_{r=1}^{\infty} |a_{n_r}| \cdot \|\alpha_{n_r}; A+(1/r)\| \leqslant \sum_{r=1}^{\infty}\left\{\frac{((A+\eta_1)\,e\rho)^{1/\rho}}{\phi(\lambda_{n_r})}\right\}^{\lambda_{n_r}} \cdot \left\{\frac{\phi(\lambda_{n_r})}{((A+\eta)\,e\rho)^{1/\rho}}\right\}^{\lambda_{n_r}}$$

$$= \sum_{r=1}^{\infty}\left(\frac{A+\eta_1}{A+\eta}\right)^{\lambda_{n_r}/\rho} \qquad \ldots(2.8)$$

and the last series is convergent. For this sequence $\{a_n\}$, $\sum_{n=1}^{\infty} a_n\alpha_n$ converges in $\chi(\rho, A, \delta)$ for each $\delta > 0$, and hence converges in χ. But, we have

$$\limsup_{n\to\infty} | a_n |^{1/\lambda_n} \phi(\lambda_n) = \limsup_{r\to\infty} \left\{ \frac{((A + \eta_1) e\rho)^{1/\rho}}{\phi(\lambda_{n_r})} \right\} \phi(\lambda_{n_r})$$

$$= ((A + \eta_1) e\rho)^{1/\rho} > (Ae\rho)^{1/\rho}.$$

This contradiction proves that (β) implies (α).

To prove that (α) implies (γ), we assume (α) is true but (γ) is not true. Hence there exists a sequence $\{a_n\}$ of complex numbers for which $a_n\alpha_n \to 0$ in χ, but $\sum_{n=1}^{\infty} a_n e_n$ does not converge in χ. This implies that

$$\limsup_{n\to\infty} | a_n |^{1/\lambda_n} \phi(\lambda_n) > (Ae\rho)^{1/\rho}. \qquad ...(2.9)$$

Hence there exists a sequence $\{n_k\}$ of positive integers and a $\lambda > 0$, such that

$$| a_n |^{1/\lambda_n} \phi(\lambda_n) \geqslant ((A + \lambda) e\rho)^{1/\rho}, \text{ for all } n = n_k. \qquad ...(2.10)$$

Now choose a positive number η, such that, $\lambda > 2\eta$. (α) being true, we can find a number $\delta = \delta(\eta)$, depending on η, such that

$$\liminf_{n\to\infty} \frac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} \geqslant \frac{1}{((A + \eta) e\rho)^{1/\rho}}.$$

Hence there exists $N = N(\eta)$, such that

$$\frac{\| \alpha_n; A + \delta \|^{1/\lambda_n}}{\phi(\lambda_n)} \geqslant \frac{1}{((A + 2\eta) e\rho)^{1/\rho}}, \text{ for all } n \geqslant N.$$

Therefore,

$$\max \| a_n\alpha_n; A + \delta \|$$

$$= \max | a_n | \cdot \| \alpha_n; A + \delta \|$$

$$\geqslant \max | a_{n_k} | \cdot \| \alpha_{n_k}; A + \delta \|$$

$$\geqslant \left\{ \frac{((A + \lambda) e\rho)^{1/\rho}}{\phi(\lambda_{n_k})} \right\}^{\lambda_{n_k}} \left\{ \frac{\phi(\lambda_{n_k})}{((A + 2\eta) e\rho)^{1/\rho}} \right\}^{\lambda_{n_k}}$$

$$> 1, \text{ for } n_k > N, \text{ as } \lambda > 2\eta.$$

This shows that $a_n\alpha_n$ does not tend to zero in $\chi(\rho, A, \delta)$ for this δ. Hence $a_n\alpha_n$ does not tend to zero in χ and this is a contradiction. Therefore we conclude that (α) implies (γ).

Now we state our main result which follows from Lemma 1 and Lemma 2.

Theorem 2 — A base $\{\alpha_n\}$ in a closed subspace χ_0 of χ is proper if and only if condition (A) and (α) are satisfied.

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