ON THE H-CONHARMONIC CURVATURE TENSOR

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In this paper, some properties of H-conharmonic curvature tensor in a Kähler manifold have been obtained.

1. INTRODUCTION

It is well known that a harmonic function is not in general transformed into harmonic function by the conformal transformation. Ishi (1957) defined conharmonic curvature tensor which is invariant under conharmonic transformations. Recently, Sinha (1972) has defined H-conharmonic curvature tensor in a Kähler manifold which may be considered as the tensor corresponding to conharmonic curvature tensor. In this paper we study the properties of H-conharmonic curvature tensor in a Kähler manifold. In treatment we consider a tensor as a vector valued or scalar valued multilinear function. The reader is referred to the papers of Mishra and Ram Hit (1971) and Yano (1965).

2. KÄHLER MANIFOLD

Let us consider a $2n$-dimensional real manifold M_{2n} of differentiability class C^{r+1} endowed with a vector valued linear function F such that

$$\bar{\bar{X}} + X = 0 \qquad \text{...(2.1)}$$

for arbitrary vector field X, where $\bar{X} \overset{def}{=} F(X)$. Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex manifold.

All the equations which follow hold for arbitrary vector fields $X, Y, Z, \ldots,$ etc.

Let M_{2n} be also endowed with the Hermitian metric tensor g :

$$g(\bar{X}, \bar{Y}) = g(X, Y). \qquad \text{...(2.2)}$$

Then M_{2n} is called an almost Hermite manifold.

Let us put

$$'F(X, Y) \overset{def}{=} g(\bar{X}, Y). \qquad \text{...(2.3)}$$

Then from (2.1), (2.2) and (2.3) we have

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$$'F(X, Y) = g(\bar{X}, Y) = -\,g(X, \bar{Y}) = -\,g(\bar{Y}, X) = -\,'F(Y, X).$$

An almost Hermite manifold for which

$$(D_X F)(Y) = 0 \qquad \ldots(2.4)$$

where D is a Riemannian connexion, is called a Kähler manifold. In the Kähler manifold the following relations hold for curvature tensor R.

$$\left.\begin{aligned}
&\text{(a)}\quad R(X, Y, \bar{Z}) = \overline{R(X, Y, Z)}\\
&\text{(b)}\quad (\operatorname{Div} R)(X, Y, Z) = (D_X \operatorname{Ric})(Y, Z) - (D_Y \operatorname{Ric})(X, Z)\\
&\text{(c)}\quad \operatorname{Ric}(X, \bar{Y}) + \operatorname{Ric}(\bar{X}, Y) = 0\\
&\text{(d)}\quad (D_X \operatorname{Ric})(Y, \bar{Z}) + (D_Y \operatorname{Ric})(Z, \bar{X}) + (D_Z \operatorname{Ric})(X, \bar{Y}) = 0
\end{aligned}\right\} \qquad \ldots(2.5)$$

where Ric is the Ricci tensor.

H-conharmonic curvature tensor S of the Kähler manifold M_{2n} is given by (Sinha 1972)

$$\begin{aligned}
S(X, Y, Z) = R(X, Y, Z) + \frac{1}{2(n+2)}\Big\{ & Y \operatorname{Ric}(X, Z) - X \operatorname{Ric}(Y, Z)\\
& + \bar{Y} \operatorname{Ric}(\bar{X}, Z) - \bar{X} \operatorname{Ric}(\bar{Y}, Z)\\
& + 2\bar{Z} \operatorname{Ric}(\bar{X}, Y) + r(Y)\, g(X, Z) - r(X)\, g(Y, Z)\\
& + r(\bar{Y})\, 'F(X, Z) - r(\bar{X})\, 'F(Y, Z) + 2r(\bar{Z})\, 'F(X, Y)\Big\}
\end{aligned}$$

where $\operatorname{Ric}(X, Y) = g(r(X), Y)$. ...(2.6)

3. H-Conharmonic Curvature Tensor

Theorem 3.1 — If a Kähler manifold with parallel H-conharmonic curvature tensor has constant scalar curvature, then it is a symmetric manifold.

Proof : Consider a Kähler manifold with parallel H-conharmonic curvature tensor. Then from (2.6) we have

$$\begin{aligned}
(D_W R)(X, Y, Z) + \frac{1}{2(n+2)}\Big\{ & Y (D_W \operatorname{Ric})(X, Z) - X (D_W \operatorname{Ric})(Y, Z)\\
& + \bar{Y}(D_W \operatorname{Ric})(\bar{X}, Z) - \bar{X}(D_W \operatorname{Ric})(\bar{Y}, Z) + 2\bar{Z}(D_W \operatorname{Ric})(\bar{X}, Y) +
\end{aligned}$$

(*equation continued on p.* 866)

$$+ g(X, Z)(D_W r)(Y) - g(Y, Z)(D_W r)(X) + 'F(X, Z)(D_W r)(\bar{Y}) - 'F(Y, Z)(D_W r)(\bar{X}) + 2'F(X, Z)(D_W r)(\bar{Z})\Big\} = 0. \qquad ...(3.1)$$

From the above equation we obtain

$$(D_Z \text{ Ric})(X, Y) = \frac{1}{4n}\{g(Y, Z)\, X\cdot R + 'F(X, Z)\, \bar{Y}\cdot R + 'F(Y, Z)\, \bar{X}\cdot R + 'F(X, \bar{Z})\, Y\cdot R + 2'F(\bar{Y}, X)\, \bar{Z}\cdot R\}$$

where we have used (2.1), (2.3), (2.4), (2.5*b*) and (2.5*d*). Thus if the scalar curvature R is constant then we have $(D_Z \text{ Ric})(X, Y) = 0$. Hence by virtue of (3.1), it follows that the Kähler manifold is symmetric.

Now, we shall define and study H-conharmonic recurrent Kähler manifold.

Definition 3.1 — A Kähler manifold M_{2n} is called an H-conharmonic recurrent Kähler manifold if

$$(D_W S)(X, Y, Z) = B(W)\, S(X, Y, Z) \qquad ...(3.2)$$

where $B(W)$ is the recurrence parameter.

Theorem 3.2 — If a Kähler manifold is a recurrent manifold with B as the parameter of recurrence, then for the same recurrence parameter it is an H-conharmonic recurrent manifold.

PROOF : From (2.6) we have

$$(D_W S)(X, Y, Z) = (D_W R)(X, Y, Z) + \frac{1}{2(n+2)}\Big\{Y(D_W \text{ Ric})(X, Z) - X(D_W \text{ Ric})(Y, Z) + g(X, Z)(D_W r)(Y) - g(Y, Z)(D_W r)(X) + (D_W \text{ Ric})(\bar{X}, Z)\bar{Y} - (D_W \text{ Ric})(\bar{Y}, Z)\bar{X} + 'F(X, Z)(D_W r)(\bar{Y}) - 'F(Y, Z)(D_W r)(\bar{X}) + 2(D_W \text{ Ric})(\bar{X}, Y)\bar{Z} + 2\,'F(X, Y)(D_W r)(\bar{Z})\Big\}. \qquad ...(3.3)$$

Let the manifold be recurrent. Then for the same recurrence parameter it is also a Ricci recurrent. For the recurrent manifold :

$$(D_W K)(X, Y, Z) = B(W)\, K(X, Y, Z). \qquad ...(3.4)$$

Using (3.4) in (3.3) we obtain

$$(D_W S)(X, Y, Z) = B(W)\Big[R(X, Y, Z) + \frac{1}{2(n+2)}\Big\{Y \operatorname{Ric}(X, Z) - X \operatorname{Ric}(Y, Z) + g(X, Z)\, r(Y) - g(Y, Z)\, r(X) + \operatorname{Ric}(\bar{X}, Z)\bar{Y} - \operatorname{Ric}(\bar{Y}, Z)\bar{X} + {}'F(X, Z)\, r(\bar{Y}) - {}'F(Y, Z)\, r(\bar{X}) + 2 \operatorname{Ric}(\bar{X}, Y)\bar{Z} + 2\, {}'F(X, Y)\, r(\bar{Z})\Big\}\Big].$$

Using (3.2) in the above equation we get

$$(D_W S)(X, Y, Z) = B(W)\, S(X, Y, Z)$$

which proves the statement.

On the other hand, the Bochner curvature tensor K is given by (Mishra and Ram Hit 1971, Tachibana 1967)

$$K(X, Y, Z) = S(X, Y, Z) - \frac{R}{4(n+1)(n+2)}\Big\{g(X, Z)\, Y - g(Y, Z)\, X + {}'F(X, Z)\, \bar{Y} - {}'F(Y, Z)\, \bar{X} + 2\, {}'F(X, Y)\, \bar{Z}\Big\} \quad ...(3.5)$$

in view of (2.6).

Put

$$s(Y, Z) = (C_1^1 S)(X, Y, Z) \quad ...(3.6)$$

where C_1^1 is the contraction. Using (3.6) in (2.6) we get

$$s(Y, Z) = -\frac{R}{2(n+2)}\, g(Y, Z). \quad ...(3.7)$$

Applying (3.7) in (3.5) we obtain

$$K(X, Y, Z) = S(X, Y, Z) + \frac{1}{2(n+1)}\Big\{s(X, Z)\, Y - s(Y, Z)\, X + s(\bar{X}, Z)\, \bar{Y} - s(\bar{Y}, Z)\, \bar{X} + 2s(\bar{X}, Y)\, \bar{Z}\Big\}. \quad ...(3.8)$$

From (3.8) we have the following theorem.

Theorem 3.3 — If a Kähler manifold is H-conharmonic recurrent, then for the same recurrence parameter it is also a Bochner recurrent manifold. Also, if in a Kähler manifold H-conharmonic curvature tensor vanishes, then the Bochner curvature tensor also vanishes.

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