ON $(f, g, u_{r|}, \lambda_{r|})$-STRUCTURE

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In studying the submanifolds of codimension 2 of an almost complex space, Yano and Okumura have defined a new structure named as (f, U, V, u, v, λ)-structure. They have also studied the case when the ambient space was almost Hermitian and in such a case they have called the submanifold to admit (f, g, u, v, λ)-structure.

In this paper, considering a submanifold of codimension $2n$ of an almost complex manifold of dimension $2m$, we have defined a new structure named as $(f, U_{r|}, u_{r|}, \lambda_{r|})$-structure, where $(r = 1, \ldots, 2n)$. In case the ambient space is an almost Hermitian manifold, the submanifold is said to admit $(f, g, u_{r|}, \lambda_{r|})$-structure. Concerning this structure we have proved that a differentiable manifold with the above structure is even dimensional. We have also proved that a submanifold of codimension $2n$ of an almost Hermitian manifold admits a $(f, g, u_{r|}, \lambda_{r|})$-structure.

1. INTRODUCTION

Submanifolds of codimension 2 of almost complex manifolds have been studied by various authors namely Ako (1967), Blair and Ludden (1969), Blair *et al.* (1970), Goldberg and Yano (1969, 1970), Okumura (1967), Yano and Ishihara (1966) and Yano and Okumura (1970). A new structure named as (f, U, V, u, v, λ)-structure has been defined and studied by Yano and Okumura (1970). When the ambient space is an almost Hermitian manifold, the submanifold is said to admit an (f, g, u, v, λ)-structure.

The purpose of the present paper is to define and study a new structure by considering a submanifold of codimension $2n$ of an almost complex manifold of dimension $2m$. We have called such a manifold as the manifold admitting $(f, U_{r|}, u_{r|}, \lambda_{r|})$-structure where $r = 1, 2, \ldots, 2n$. We have also considered the case when the ambient space is an almost Hermitian manifold and in such a case we have called the structure an $(f, g, u_{r|}, \lambda_{r|})$-structure. We have studied some properties of these manifolds and obtained some interesting results.

2. $(f, U_{r|}, u_{r|}, \lambda_{r|})$-STRUCTURE

Let M be an m-dimensional differentiable manifold of class C^∞ and let there exist on M a tensor field of type (1, 1), vector fields $U_{r|}$, $(r = 1, 2, \ldots, 2n)$, 1-forms $u_{r|}$ and r functions $\lambda_{r|}$ satisfying the following conditions :

$$f^2X = -X + \sum_r u_{r|}(X)U_{r|}, \quad \ldots(2.1)$$

for any vector field X

$$u_{l|} \circ f = \lambda_{l|}\, u_{l+1|}, fU_{l|} = -\lambda_{l|}\, U_{l+1|}, l = 1, 2, \ldots, n \quad \ldots(2.2)$$

$$u_{p|} \circ f = -\lambda_{p|}\, u_{p-1|}, fU_{p|} = \lambda_{p|}\, U_{p-1|}, p = n+1, \ldots, 2n \quad \ldots(2.3)$$

where 1-forms $u_{r|} \circ f$ are defined by

$$(u_{r|} \circ f)(X) = u_{r|}(fX)$$

and

$$u_{r|}(U_{k|}) = 1 - \lambda^2_{r|} \quad (r = k)$$

$$= 0 \qquad (r \neq k). \quad \ldots(2.4)$$

Definition 2.1 — The manifold M which satisfies all the above conditions is called a manifold having $(f, U_{r|}, u_{r|}, \lambda_{r|})$-structure.

Theorem 2.1 — A differentiable manifold with $(f, U_{r|}, u_{r|}, \lambda_{r|})$-structure is of even dimension.

PROOF : Let P be a point of M which is such that for it $\lambda^2_{r|} \neq 1$, then from (2.4) we have $U_{r|} \neq 0$ at P. The vectors $U_{r|}$ are r linearly independent vectors, for if there are r numbers $a_{r|}$, such that

$$\sum_r a_{r|}\, U_{r|} = 0 \quad \ldots(2.5)$$

then

$$\sum_r u_{r|}(a_{r|}\, U_{r|}) = \sum_r a_{r|}\, u_{r|}(U_{r|}) = \sum_r a_{r|}(1 - \lambda^2_{r|}) = 0$$

which implies that $a_{r|} = 0$.

Thus $U_{r|}$ being linearly independent at P we can choose m linearly independent vectors $X_{r|} = U_{r|}, X_{r+1|}, \ldots, X_{m|}$, which span the tangent space $T_P(M)$ of M at P and such that $u_{r|}(X_{\alpha|}) = 0$, for $\alpha = r+1, \ldots, m$. Consequently from (2.1) we have

$$f^2X_{\alpha|} = -X_{\alpha|}, \alpha = r+1, \ldots, m,$$

which shows that f is an almost complex structure in the subspace V_P of $T_P(M)$ at P spanned by $X_{r+1|}, \ldots, X_{m|}$ and that V_P is even dimensional. Hence $T_P(M)$ is also even dimensional.

Next let P be a point of M at which $\lambda^2_{r|} = 1$, then from (2.4) we have $u_{r|}(U_{r|}) = 0$. From equations (2.2) and (2.3) we observe that when one of the u's is not zero others are also not zero and when one of the u's is zero others are also zero.

We first consider the case in which $u_{r|} \neq 0$. In this case it can be easily proved that u's are linearly independent, therefore we can choose m linearly independent covectors $w_{r|} = u_{r|}, w_{r+1|}, \ldots, w_{m|}$ which span the cotangent space ${}^{C}T_P(M)$ of M at P. Denoting the dual basis by $(X_{1|}, \ldots, X_{m|})$, if $U_{r|}$ are linearly independent at P, we can assume that $X_{m-r+1|} = U_{1|}, X_{m-r+2|} = U_{2|}, \ldots, X_{m|} = U_{r|}$.
Thus we have

$$f^2 X_{\alpha|} = -X_{\alpha|} + \sum_r u_{r|}(X_{\alpha|})\, U_{r|} = -X_{\alpha|},\ \alpha = r+1, \ldots, m,$$

which shows that f is an almost complex structure in the subspace V_P of $T_P(M)$ at P spanned by $X_{r+1|}, \ldots, X_{m|}$ and that V_P is even dimensional and consequently $T_P(M)$ is also even dimensional.

If $U_{r|}$ are linearly dependent, then it can be easily proved that $U_{r|} = 0$ for every r. Thus $T_P(M)$ is even dimensional because from (2.1) we obtain $f^2X = -X$ for any vector X in $T_P(M)$.

The case in which $u_{r|} = 0$, by virtue of (2.1) immediately implies that $f^2X = -X$ and hence $T_P(M)$ is even dimensional.

This completes the proof of Theorem.

We assume that in M with $(f, U_{r|}, u_{r|}, \lambda_{r|})$-structure there exists a positive definite Riemannian metric g such that

$$g(U_{r|}, X) = u_{r|}(X) \qquad \ldots(2.6)$$

and

$$g(fX, fY) = g(X, Y) - \sum_r u_{r|}(X)\, u_{r|}(Y), \qquad \ldots(2.7)$$

for any vector field X and Y of M. We call such a structure a metric $(f, U_{r|}, u_{r|}, \lambda_{r|})$-structure and denote it by $(f, g, u_{r|}, \lambda_{r|})$-structure. Now we prove the following theorm.

Theorem (2.2) — Let ω be a tensor field of type (0, 2) of M defined by

$$\omega(X, Y) = g(fX, Y) \qquad \ldots(2.8)$$

for any vector field X and Y of M then we have

$$\omega(fX, fY) = -\,\omega(fY, fX). \qquad \ldots(2.9)$$

PROOF : From (2.8) we have

$$\omega(fX, fY) = g(f(fX), fY)$$

which by virtue of (2.7) implies

$$\omega(fX, fY) = g(fX, Y) - \sum_r u_{r|}(fX)\, u_{r|}(Y).$$

Using (2.2) and (2.3) in the above we obtain

$$\omega(fX, fY) = \omega(X, Y) - \sum_{l=1}^{n} \lambda_{l|}\, u_{l+1|}(X)\, u_{l|}(Y)$$

$$+ \sum_{p=n+1}^{2n} \lambda_{p|}\, u_{p-1|}(X)\, u_{p|}(Y). \qquad \ldots(2.10)$$

On the other hand we have

$$\omega(fY, fX) = g(f^2Y, fX) = g(-Y + \sum_r u_r(Y)\, U_{r|}, fX)$$

$$= -g(Y, fX) + \sum_r u_{r|}(Y)\, u_{r|}(fX)$$

which can be written as

$$\omega(fY, fX) = -\,\omega(X, Y) + \sum_{l=1}^{n} \lambda_{l|}\, u_{l+1|}(X)\, u_{l|}(Y)$$

$$- \sum_{p=n+1}^{2n} \lambda_{p|}\, u_{p-1|}(X)\, u_{p|}(Y). \qquad \ldots(2.11)$$

Comparing (2.10) and (2.11) we obtain

$$\omega(fX, fY) = -\,\omega(fY, fX).$$

(Proved)

3. Submanifolds of Codimension $2n$ of an Almost Hermitian Manifold

We shall first of all prove the following theorem :

Theorem 3.1 — A submanifold of codimension $2n$ of an almost Hermitian manifold admits an $(f, g, u_{r|}, \lambda_{r|})$-structure.

PROOF : Let M^* be a $2m$-dimensional almost Hermitian manifold covered by a system of coordinate neighbourhoods $\{U^*; y^\alpha\}$ ($\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, 2m$), and let $(F^\beta_\gamma, G_{\delta\gamma})$ be the almost Hermitian structure such that F^β_γ is the almost complex structure satisfying (Yano 1965) :

$$F^{\beta}_{\gamma} F^{\gamma}_{\alpha} = - \delta^{\beta}_{\alpha} \qquad ...(3.1)$$

and

$$G_{\gamma\beta} F^{\gamma}_{\alpha} F^{\beta}_{\delta} = G_{\alpha\delta} \qquad ...(3.2)$$

where $G_{\alpha\delta}$ is a Riemannian metric.

Let M be a $2m - 2n$ dimensional differentiable manifold which is covered by a system of coordinate neighbourhoods $\{U; x^i\}$, $i, j, k, ... = 1, 2, ..., 2(m - n)$ and let M be differentiably immersed in M^* as a submanifold of codimension $2n$ by the equations

$$y^{\alpha} = y^{\alpha} (x^i).$$

If we put $\partial_i y^{\alpha} = B^{\alpha}_i$ where $\partial_i = \partial/\partial x^i$, then for each fixed i, B^{α}_i is a local vector field of M^* tangent to M and the vectors B^{α}_i are linearly independent in each coordinate neighbourhood. Also for each fixed α, B^{α}_i is a local 1-form of M.

Let $N^{\alpha}_{r|}$ be $2n$ mutually orthogonal unit vectors of M^* normal to M in such a way that $r = 1, 2, ..., 2n$ and $2m$ vectors B^{α}_i and $N^{\alpha}_{r|}$ give the positive orientation of M.

The transforms $F^{\alpha}_{\beta} \; B^{\beta}_i$ of B^{β}_i by F^{α}_{β} can be expressed as a linear combination of B^{β}_i and $N^{\alpha}_{r|}$ in the following form :

$$F^{\beta}_{\alpha} B^{\alpha}_i = f^h_i B^{\beta}_h + \sum_{r=1}^{2n} u_{r|\,i} N^{\beta}_{r|} \qquad ...(3.3)$$

where f^h_i is a tensor field of type (1, 1) and $u_{r|\,i}$ are 1-forms of M. Similarly the transforms $F^{\alpha}_{\beta} \; N^{\beta}_{r|}$ are given by

$$F^{\alpha}_{\beta} \; N^{\beta}_{l|} = - u^i_{l|} \; B^{\alpha}_i + \sum_{p=n+1}^{2n} \lambda_{p|} N^{\alpha}_{p|}, \; l = 1, 2, ..., n \qquad ...(3.4a)$$

and

$$F^{\alpha}_{\beta} \; N^{\beta}_{p|} = - u^i_{p|} \; B^{\alpha}_i - \sum_{l=1}^{n} \lambda_{l|} \; N^{\alpha}_{l|}, p = n + 1, ..., 2n \qquad ...(3.4b)$$

where $u^i_{r|} = g^{ti} u_{r|t}$ and g_{ji} is the Riemannian metric on M induced from that of M^*. Also we have

$$g_{ji} = G_{\alpha\beta} B^\alpha_j B^\beta_i . \qquad \ldots(3.5)$$

Applying F^δ_β to (3.3) and using eqns. (3.3), (3.4a, b) we obtain on simplification

$$f^h_i f^j_h = - \delta^j_i + \sum_{r=1}^{2n} u_{r|i} u^j_{r|} \qquad \ldots(3.6)$$

and

$$f^i_j u_{l|i} = \lambda_{l|} u_{l+1|j}, f^i_j u_{p|i} = - \lambda_{p|} u_{p-1|j} \qquad \ldots(3.7)$$

$$(l = 1, \ldots, n; p = n + 1, \ldots, 2n).$$

Applying F^δ_α to (3.4a, b) and using eqns. (3.3), (3.4a, b) we obtain

$$f^h_i u^i_{l|} = - \lambda_{l|} u^h_{l+1|} , f^h_i u^i_{p|} = \lambda_{p|} u^h_{p-1|} \qquad \ldots(3.8)$$

and

$$u^i_{l|} u_{p|i} = 0, u^i_{r|} u_{r|i} = 1 - \lambda^2_{r|} . \qquad \ldots(3.9)$$

On the other hand from (3.2) we have

$$G_{\gamma\beta} F^\gamma_\alpha F^\beta_\delta B^\delta_j B^\delta_i = G_{\alpha\delta} B^\alpha_j B^\delta_i$$

which implies

$$g_{ts} f^t_j f^s_i = g_{ji} - \sum_{r=1}^{2n} u_{r|i} u_{r|j}. \qquad \ldots(3.10)$$

Equations (3.6), (3.7), (3.8), (3.9) and (3.10) show that a submanifold of co-dimension $2n$ of an almost Hermitian manifold admits an (f, g, $u_{r|}$, $\lambda_{r|}$)-structure.

(Proved).

The Van der Waerden-Bortolotti covariant derivative of B^α_i can be expressed as (Yano 1965).

$$\nabla_j B^\alpha_i = \partial_j B^\alpha_i + \left\{ {\alpha \atop \beta\gamma} \right\} B^\beta_j B^\gamma_i - B^\alpha_h \left\{ {h \atop ji} \right\} \qquad \ldots(3.11)$$

where ∇_j and $\left\{ {h \atop ji} \right\}$ respectively denote the covariant derivative and Christoffel symbol corresponding to g_{ji} and $\left\{ {\alpha \atop \beta\gamma} \right\}$ is the Christoffel symbol corresponding to $G_{\beta\gamma}$.

Since $\nabla_j B_i^\alpha$ is orthogonal to M it can be expressed as

$$\nabla_j B_i^\alpha = \sum_{r=1}^{2n} H_{r\,|\,ji}\, N_{r\,|}^{\alpha} \qquad ...(3.12)$$

where $H_{r\,|\,ji}$ are the second fundamental tensors of M. Equation (3.12) is analogous to Gauss-Equation.

If

$$\nabla_j N_{r\,|}^{\alpha} = \partial_j N_{r\,|}^{\alpha} + \left\{ {\alpha \atop \beta\gamma} \right\} B_j^\beta\, N_{r\,|}^{\alpha}$$

and $H_{r\,|\,j}^{\;i} = H_{r\,|\,jk}\, g^{ki}$, then the equations of Weingarten are given by

$$\nabla_j N_{l\,|}^{\alpha} = -\sum_{l=1}^{n} H_{l\,|\,j}^{\;i}\, B_i^\alpha + \sum_{p=n+1}^{2n} R_{p\,|\,j}\, N_{p\,|}^{\alpha} \qquad ...(3.13a)$$

and

$$\nabla_j N_{p\,|}^{\alpha} = -\sum_{p=n+1}^{2n} H_{p\,|\,j}^{\;i} B_i^\alpha - \sum_{l=1}^{n} R_{l\,|\,j}\, N_{l\,|}^{\alpha} \qquad ...(3.13b)$$

where $R_{r\,|\,j}$ $(r = 1, ..., 2n)$ are the third fundamental tensors.

From eqns. (3.13a, b) the connexions induced in the normal bundle can be given by

$$'\nabla_j N_{l\,|}^{\alpha} = \sum_{p=n+1}^{2n} R_{p\,|\,j}\, N_{p\,|}^{\alpha} \qquad ...(3.14a)$$

and

$$'\nabla_j N_{p\,|}^{\alpha} = -\sum_{l=1}^{n} R_{l\,|\,j} N_{l\,|}^{\alpha}. \qquad ...(3.14b)$$

If this induced connexion is flat we can choose $N_{r\,|}^{\alpha}$ in such a way that $R_{r\,|\,j} = 0$.

Differentiating eqn. (3.3) covariantly along M and using eqns. (3.12) and (3.13a, b) we obtain

$$\left(\nabla_\gamma F^\beta_\alpha \right) B^\gamma_j B^\alpha_i + \sum_{r=1}^{2n} F^\beta_\alpha H_{r\,|\,ji} N^\alpha_{r\,|}$$

$$= \left(\nabla_j f^h_i \right) B^\beta_h + f^t_i \sum_{r=1}^{2n} H_{r\,|\,jt} N^\beta_{r\,|} + \sum_{r=1}^{2n} (\nabla_j u_{r\,|\,i}) N^\beta_{r\,|}$$

$$- \sum_{l=1}^{n} u_{l\,|\,i} \left(\sum_{l=1}^{n} H^t_{l\,|\,j} B^\beta_t - \sum_{p=n+1}^{2n} R_{p\,|\,j} N^\beta_{p\,|} \right)$$

$$- \sum_{p=n+1}^{2n} u_{p\,|\,i} \left(\sum_{p=n+1}^{2n} H^t_{p\,|\,j} B^\beta_t + \sum_{l=1}^{n} R_{l\,|\,j} N^\beta_{l\,|} \right)$$

which on further simplification leads to

$$\left(\nabla_\gamma F^\beta_\alpha \right) B^\gamma_j B^\alpha_i + \sum_{l=1}^{n} \left\{ - u^t_{l\,|} B^\beta_t + \sum_{p=n+1}^{2n} \lambda_{p\,|} N^\beta_{p\,|} \right\} H_{l\,|\,ji}$$

$$+ \sum_{p=n+1}^{2n} H_{p\,|\,ji} \left\{ - u^t_{p\,|} B^\beta_t - \sum_{l=1}^{n} \lambda_{l\,|} N^\beta_{l\,|} \right\}$$

$$= \left(\nabla_j f^h_i - \sum_{r=1}^{2n} u_{r\,|\,i} H^h_{r\,|\,j} \right) B^\beta_h + \sum_{r=1}^{2n} (\nabla_j u_{r\,|\,i} + f^t_i H_{r\,|\,jt}) N^\beta_{r\,|}$$

$$+ \sum_{l=1}^{n} u_{l\,|\,i} \sum_{p=n+1}^{2n} R_{p\,|\,j} N^\beta_{p\,|} - \sum_{p=n+1}^{2n} u_{p\,|\,i} \sum_{l=1}^{n} R_{l\,|\,j} N^\beta_{l\,|}.$$

If M is a Kählerian manifold we have $\nabla_\alpha F^\gamma_\beta = 0$, then we have

$$\nabla_j f^h_i = \sum_{r=1}^{2n} (u_{r\,|\,i} H^h_{r\,|\,j} - u^h_{r\,|} H_{r\,|\,ji}) \qquad \text{...(3.15}a\text{)}$$

and

$$\nabla_j u_{l\,|\,i} = - H_{l\,|\,jt} f^t_i + \sum_{p=n+1}^{2n} (u_{p\,|\,i} R_{l\,|\,j} - H_{p\,|\,ji} \lambda_{l\,|}) \qquad \text{...(3.15}b\text{)}$$

and

$$\nabla_j u_{p|i} = -H_{p|jt} f_i^t - \sum_{l=1}^{n} (u_{l|i} R_{p|j} - H_{l|ji} \lambda_{p|}). \quad ...(3.15c)$$

Now we define a tensor S_{ji}^h as follows :

$$S_{ji}^h \overset{def}{=} N_{ji}^h + \sum_{r=1}^{2n} (\nabla_j u_{r|i} - \nabla_i u_{r|j}) u_{r|}^h \quad ...(3.16)$$

where N_{ji}^h is the Niejenhuis tensor.

By virtue of equations (3.15*a*, *b*, *c*) equation (3.16) takes the form

$$S_{ji}^h = \sum_{r=1}^{2n} \left\{ \left(f_j^t H_{r|t}^h - H_{r|j}^t f_t^h \right) u_{r|i} - \left(f_i^t H_{r|t}^h - H_{r|i}^t f_t^h \right) u_{r|j} \right\}$$

$$+ \sum_{l=1}^{n} u_{l|}^h \sum_{p=n+1}^{2n} (u_{p|i} R_{l|j} - u_{p|j} R_{l|i})$$

$$- \sum_{p=n+1}^{2n} u_{p|}^h \sum_{l=1}^{n} (R_{p|j} u_{l|i} - R_{p|i} u_{l|j}). \quad ...(3.17)$$

Definition 3.1 — We define a manifold for which $S_{ji}^h = 0$ as a manifold admitting a normal $(f, g, u_{r|}, \lambda_{r|})$-structure.

From definition (3.1) and eqn. (3.17) we can easily have the following.

Theorem 3.2 — The necessary and sufficient condition for a submanifold of codimension $2n$ of a Kählerian manifold whose connection induced in the normal bundle is flat, to admit a normal $(f, g, u_{r|}, \lambda_{r|})$-structure is that f commutes with $H_{r|}$.

For a totally umbilical submanifold whose connection induced in the normal bundle is flat, we have for suitably chosen normals $N_{r|}$,

$$H_{r|ji} = H_{r|} g_{ji}, \; R_{r|j} = 0.$$

Consequently eqns. (3.15*b*, *c*) become

$$\nabla_j u_{l|i} = - H_{l|} f_{ji} - \sum_{p=n+1}^{2n} \lambda_{l|} H_{p|} g_{ji} \quad ...(3.18a)$$

and

$$\nabla_j u_{p|i} = - H_{p|} f_{ji} + \sum_{l=1}^{n} \lambda_{p|} H_{l|} g_{ji} \qquad \text{...(3.18}b\text{)}$$

respectively.

From these equations we obtain

$$\nabla_j u_{l|i} + \nabla_i u_{l|j} = -2 \sum_{p=n+1}^{2n} \lambda_{l|} H_{p|} g_{ji} \qquad \text{...(3.19}a\text{)}$$

and

$$\nabla_j u_{p|i} + \nabla_i u_{p|j} = 2 \sum_{l=1}^{n} \lambda_{p|} H_{l|} g_{ji} \qquad \text{...(3.19}b\text{)}$$

and hence the following theorem :

Theorem 3.3 — For a totally umbilical submanifold whose connexion induced in the normal bundle is flat the vectors $u^h_{r|}$ define infinitesimal conformal transformations in M.

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