ON THE STRUCTURE OF UNIFORMLY ROTATING POLYTROPES

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(Communicated by F. C. Auluck, F.N.A.)

(*Received* 22 *May* 1973)

The equilibrium structure of uniformly rotating polytropes is investigated using a simple, analytic, iterative technique with starting point, the spherical structure. The iteration converges rapidly for $n > 1$ and at each stage no more numerical work is involved than that required to solve the spherical problem. Critical configurations are also discussed and results obtained compared with previous work in this field.

1. INTRODUCTION

The study of the structure of rapidly rotating polytropes has been advanced considerably in recent years following the pioneer work of Chandrasekhar (1933). The classical analyses of MacLaurin (1742), Jacobi (1834), Riemann (1860) and Poincaré (1891) for rotating liquid sphere exemplify the inherent topological difficulty encountered in such considerations : that Poisson's equation must be integrated over a domain, the discovery of which constitutes part of the problem.

James' (1964) and Stoekly's (1965) numerical integrations, although extremely valuable as a guide are limited to $n < 3$ and instil no confidence in tackling the much more complex problem of rotating real stars. The variational approach adopted by Roberts (1963a, b) and Hurley and Roberts (1964) gives a very poor representation of the heavily-distorted surface layers since equipotential surfaces are taken to be spheroids, necessarily inexact (even in the average sense) at critical rotation.

The double-approximation technique devised by Monaghan and Roxburgh (1965) for polytropes is readily extended to real stars (Roxburgh and Strittmatter 1965, Roxburgh *et al.* 1965, Faulkner *et al.* 1968) since only minor modifications are required to programs originally written to compute the spherical structure. However, no attempt is made to solve the coupled thermal and mechanical equations; rather it is assumed that uniform rotation is maintained even in the presence of meridional currents by a weak magnetic field or viscosity. Our interest in rotating polytropes arose in this context.

It is noticeable in the formulation of the polytrope problem that the Chandrasekhar-based methods (Monaghan and Roxburgh 1965, Anand 1968, Martin 1970) require identical equations to be integrated at all rotation speeds, there being no simplification for slow rotation. The inability of the extended methods to follow the evolution of the angular velocity field of real stars is traced back to this fact. We thus present here a simple, analytic, iterative approach to the polytrope problem which might be loosely termed an "iterating matched-asymptotic expansion" sacrificing accuracy for ease of application if necessary. We retain the original differential equations rather than their more conventional integral representations as used in other iterative schemes (Ostriker and Mark 1968, Mark 1968, etc.) for comparative purposes. The iteration converges rapidly for $n > 1$ and no more numerical effort is required than that needed to solve the spherical problem.

2. Formulation of the Problem

The structure equations of a uniformly-rotating polytrope of index n are, with the usual notation,

$$\frac{\nabla P}{\rho} = -\nabla\Phi + \Omega^2\tilde{\omega} \qquad \text{...(2.1)}$$

$$P = K\rho^{1+(1/n)} \qquad \text{...(2.2)}$$

$$\nabla^2\Phi = 4\pi G\rho. \qquad \text{...(2.3)}$$

Although in general $K = K(\Omega)$ we shall adopt the standard procedure of considering K a constant independent of the angular velocity; this just introduces a scaling factor to the linear dimensions. If r denotes the radial coordinate, μ the cosine of the colatitude and ρ_c the central density, we may conveniently define the polytropic variables ξ, σ, ϕ and α by the relations

$$\rho = \rho_c\sigma^n;\ \Phi = -K(n+1)\,\rho_c^{1/n}\,\phi;\ r = a\xi;\ \alpha = \frac{\Omega^2}{2\pi G\rho_c} \qquad \text{...(2.4)}$$

where $a^2 = \dfrac{K(n+1)}{4\pi G}\rho_c^{1/(n-1)}$. Combining (2.1), (2.2) and integrating gives

$$\sigma = \phi + \frac{\alpha\xi^2}{6}\Big(1 - P_2(\mu)\Big) + \text{constant} \qquad \text{...(2.5)}$$

while (2.3) yields

$$\nabla^2_{\xi,\mu}\phi = \frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\phi}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\phi}{\partial\mu}\right) = -\sigma^n \qquad \text{...(2.6)}$$

where $P_2(\mu)$ denotes the second Legendre Polynomial. From (2.5) and (2.6) we derive the fundamental equation :

$$\nabla^2_{\xi,\mu}\sigma = -\sigma^n + \alpha \qquad \text{...(2.7)}$$

with boundary conditions,

$$\sigma = 1, \nabla\sigma = 0 \text{ at } \xi = 0. \qquad \text{...(2.8)}$$

The spherical analogue of (2.7) is the well-known Emden equation :

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

the solutions of which are known in tabular form. The Emden sphere is the solution of $\theta(\xi_1) = 0$ and represents the surface of the non-rotating configuration. In the subsequent analysis we assume $\alpha \ll 1$.

3. Method of Solution

The exponent $n > 1$ entering the r.h.s. of (2.7) invites us to set up an iterative procedure :

$$\nabla^2\sigma_{k+1} = -\sigma_k^n + \alpha \qquad \text{...(3.1)}$$

or more explicitly,

$$\nabla^2\phi_{k+1} = -\sigma_k^n \qquad \text{...(3.1}a\text{)}$$

$$\sigma_{k+1} = \phi_{k+1} + \frac{\alpha\xi^2}{6}\left(1 - P_2\right) + \text{constant.} \qquad \text{...(3.1}b\text{)}$$

The most natural starting point for the iteration is the spherically-symmetric distribution. We thus define

$$\left.\begin{array}{ll} \textit{Inner}: \sigma_{-1} = \theta(\xi) & \text{for } 0 \leqslant \xi \leqslant \xi_1 \\ \textit{Outer}: \sigma_{-1} = 0 & \text{for } \xi_1 \leqslant \xi \end{array}\right\} \qquad \text{...(3.2)}$$

where $\theta(\xi)$ satisfies Emden's equation

$$\nabla^2\theta = -\theta^n. \qquad \text{...(3.3)}$$

Substituting (3.2) into the r.h.s. of (3.1) and integrating gives, for the inner region,

$$\sigma_{0i} = \theta + \frac{\alpha\xi^2}{6}(1 - P_2) + \alpha\sum_{m=1}^{\infty} a_{2m}\xi^{2m}P_{2m}(\mu) \qquad \text{...(3.4)}$$

where the a_{2m} are arbitrary constants. An arbitrary solution of Laplace's equation is introdued into the second term for consistency with (3.1b) while the exclusion of the $m = 0$ term ensures that σ_0 satisfies the central conditions (2.8). Only even harmonics are included due to the symmetry about the equatorial plane, the factor α being introduced into the last terms to illustrate that $\sigma_0 \to \theta$ as $\alpha \to 0$. With standard conditions at infinity, the outer solution is

$$\sigma_{0e} = a_0 + \frac{b_0}{\xi} + \frac{\alpha\xi^2}{6}\left(1 - P_2\right) + \alpha \sum_{m=1}^{\infty} \frac{b_{2m}}{\xi^{2m+1}} P_{2m}(\mu). \qquad \text{...(3.5)}$$

As $\alpha \to 0$ we must recover sphericity, hence the factor α in the last terms.

We now match the interior and exterior solutions across the (hypothetical) surface $\xi = \xi_1$ ensuring the continuity of σ_0 and its gradient. Through (3.1*b*) this then implies that ϕ_0 and $\nabla\phi_0$ are also continuous. We find that

$$a_0 = \xi_1\theta_1;\ b_0 = -\ \xi_1^2\theta_1;\ a_{2m} = b_{2m} = 0 \qquad \text{for } m \geqslant 1 \qquad \text{...(3.6)}$$

where $\theta_1 = \theta'(\xi_1)$. The zero order solution is then explicitly :

$$\textit{Inner} : \sigma_{0i} = \theta + \frac{\alpha\xi^2}{6}\left(1 - P_2\right) \qquad \text{for } 0 \leqslant \xi \leqslant \xi_1 \quad \text{..(3.7}a\text{)}$$

$$\textit{Outer} : \sigma_{0e} = \xi_1\theta_1(1 - \xi_1/\xi) + \frac{\alpha\xi^2}{6}\left(1 - P_2\right) \qquad \text{for } \xi_1 \leqslant \xi. \quad \text{...(3.7}b\text{)}$$

The surface of the configuration is given by $\sigma_{0e}(\xi, \mu) = 0$ and from (3.7*b*) we see immediately that to this approximation the polar radius is ξ_1, for all rotation speeds. There is a critical value of α for which the effective gravity vanishes at the surface of the polytrope at a distance ξ_e along the equatorial plane. The critical parameters α_c ξ_e are determined from the equations

$$\sigma_{0e} = \frac{\partial\sigma_{0e}}{\partial\xi} = 0 \text{ at } \mu = 0. \qquad \text{...(3.8)}$$

From (3.7*b*) we find that

$$\xi_e = \frac{3}{2}\,\xi_1;\ \alpha_c = -\frac{16}{27}\,\frac{\theta_1}{\xi_1}. \qquad \text{...(3.9)}$$

Details are given in Table II for various values of n.

The next iteration follows logically as

$$\textit{Inner} : \nabla^2\sigma_1 = -\ \sigma_{0i}^n + \alpha \quad \text{for } 0 \leqslant \xi \leqslant \xi_1$$

$$\textit{Mid} : \quad = -\ \sigma_{0e}^n + \alpha \quad \text{for } \xi_1 \leqslant \xi \leqslant \xi_s$$

$$\textit{Outer} : \quad = \alpha \quad \text{for } \xi_s \leqslant \xi$$

where ξ_s is the zeroeth approximation to the surface, i.e. the solution of $\sigma_{0e}(\xi_s, \mu) = 0$. We do not in fact follow the iteration in this way but rather construct a Roche envelope onto the first order interior solution and reconsider surface layers once satisfactory convergence has been realised. The governing equations thus reduce to

$$\nabla^2\sigma_1 = -\left[\theta + \frac{\alpha\xi^2}{6}\left(1 - P_2\right)\right]^n + \alpha \text{ for } 0 \leqslant \xi \leqslant \xi_1$$
$$= \alpha \qquad \text{for } \xi_1 \leqslant \xi.$$

We expand the exponent assuming θ to be the dominant term and, for simplicity, retain terms up to order α only. This gives

$$\nabla^2\sigma_1 = -\theta^n - \frac{n\alpha}{6}\xi^2\theta^{n-1}(1 - P_2) + \alpha, \qquad 0 \leqslant \xi \leqslant \xi_1 \qquad \text{...(3.11)}$$

and thence,

$$\sigma_{1i} = \sigma_{0i} + \alpha n(\psi_1(\xi) - \psi_2(\xi)\, P_2) + \alpha \sum_{m=1}^{\infty} a_{2m}\xi^{2m}P_{2m} \qquad \text{...(3.12)}$$

where ψ_1, and ψ_2 are, respectively, the particular solutions of

$$\psi_1'' + \frac{2}{\xi}\psi_1' = -\frac{\xi^2}{6}\theta^{n-1}$$
$$\psi_2'' + \frac{2}{\xi}\psi_2' - \frac{6}{\xi^2}\psi_2 = -\frac{\xi^2}{6}\theta^{n-1} \qquad \text{...(3.13)}$$

which satisfy $\psi_1 = \psi_2 = \psi_1' = \psi_2' = 0$ at $\xi = 0$. These are readily seen to be

$$\psi_1 = -\frac{1}{6}\int_0^\xi \frac{dx}{x^2}\int_0^x y^4\theta^{n-1}(y)\, dy; \; \psi_2 = -\frac{\xi^2}{6}\int_0^\xi \frac{dx}{x^6}\int_0^x y^6\theta^{n-1}(y)\, dy. \quad \text{...(3.14)}$$

Reversing the order of integration we can integrate once to obtain

$$\left.\begin{aligned} \psi_1 &= -\frac{1}{6}\left(I_1 - I_2\right) \\ \psi_2 &= -\frac{1}{30}\left(I_3 - I_4\right) \end{aligned}\right\} \qquad \text{...(3.15)}$$

where

$$I_1 = \int_0^\xi y^3\theta^{n-1}dy; \quad I_2 = \frac{1}{\xi}\int_0^\xi y^4\theta^{n-1}dy$$

$$I_3 = \xi^2\int_0^\xi y\theta^{n-1}dy; \quad I_4 = \frac{1}{\xi^3}\int_0^\xi y^6\theta^{n-1}dy.$$

Each of the four integrals may be computed along with the spherical structure, and indeed would be analytic if θ was. Matching (3.12) with the exterior solution (2.13) across the Emden sphere fixes σ_1 :

$$\sigma_{1i} = \sigma_{0i} - \frac{\alpha n}{6}\left(I_1 - I_2\right) + \frac{\alpha n}{30}\left(I_3 - I_{31}\frac{\xi^2}{\xi_1^2} - I_4\right)P_2 \quad \text{for } 0 \leqslant \xi \leqslant \xi_1 \qquad \ldots(3.16)$$

$$\sigma_{1e} = \sigma_{0e} - \frac{\alpha n}{6}\left(I_{11} - I_{21}\frac{\xi_1}{\xi}\right) - \frac{\alpha n}{30} I_{41}\frac{\xi_1^3}{\xi^3} P_2 \quad \text{for } \xi_1 \leqslant \xi \qquad \ldots(3.17)$$

where $I_{11} = I_1(\xi_1)$ etc. The critical parameters $\eta_c = \xi_e/\xi_1$ and α_c are determined from the equations

$$10n(I_{11} - I_{21}) - 15\xi_1^2\left(3\eta_c^2 - 2\eta_c^3\right) - nI_{41}\left(\frac{3}{\eta_c^2} - \frac{2}{\eta_c^3}\right) = 0 \qquad \ldots(3.18)$$

$$\frac{1}{\alpha_c} = \frac{1}{\xi_1\theta_1}\left(\frac{3}{4}\xi_1^2\eta_c^2 - \frac{nI_{11}}{6} - \frac{n}{30}\frac{I_{41}}{\eta_c^3}\right).$$

Details are given in Table II.

4. The Neglected Terms

In (3.11) we neglected the term

$$[\theta + \alpha\tau]^n - \theta^n - n\alpha\theta^{n-1}\tau$$

where, for brevity, we have written $\tau = \frac{\xi^2}{6}\left(1 - P_2\right)$. This has a maximum on the equator at $\xi = \xi_1$, where its value is

$$\epsilon_1 = \left(\frac{\alpha\xi_1^2}{4}\right)^n.$$

The maximum value of the neglected terms beyond the Emden sphere is given by the same quantity. For critical rotation the results are seen in Table I.

Table I

n	1.5	2.0	2.5	3.0	3.5	4.0
ξ_1	3.654	4.353	5.417	6.901	9.536	14.972
ϵ_1	0.09	0.01	0.004	0.0005	0.000004	0.000001
ϵ_2	0.012	0.0062	0.0015	0.0002	—	—

With n non-integral we recognise an inherent mathematical difficulty which is not encountered for integral n. With n an integer the power series expansion of $[\theta + \alpha\tau]^n$ is finite and there is no worry about convergence. For non-integral n

however the series is infinite and converges only for $\alpha\tau/\theta \leqslant 1$ i.e. only interior to the surface

$$\theta(\xi) = \frac{\alpha\xi^2}{6}\left(1 - P_2\right)$$

and for α non-zero this lies inside the Emden sphere as shown schematically in Fig. 1. If we disregard the above discussion and match across $\xi = \xi_1$, we introduce an error of fixed amount and our interior solution will not converge to the exact solution no matter how many terms we retain in the expansion of $[\theta + \alpha\tau]^n$. For integral n we may, in principle, take account of every term of the expansion and expect convergence to the exact interior solution. Surface layers are discussed later.

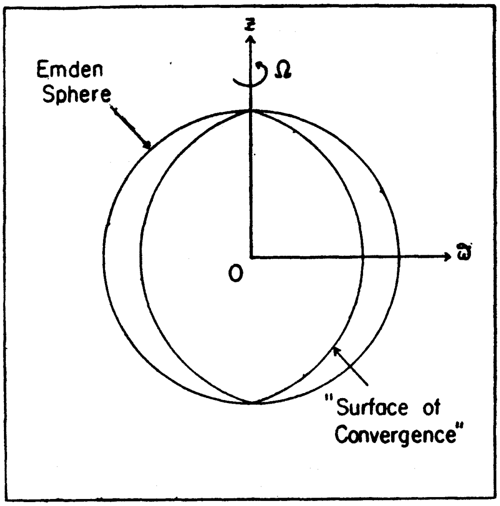

FIG. 1.

5. ITERATION CONTINUED

We now continue the iteration to second order interior to the Emden sphere keeping the zero-order exterior approximation and again retaining only first-order terms in α :

$$\nabla^2\sigma_2 = -\sigma_{1i}^n + \alpha, \quad 0 \leqslant \xi \leqslant \xi_1$$

$$= \alpha, \quad \xi_1 \leqslant \xi.$$

The results of matching as before are

$$\sigma_{2i} = \sigma_{1i} + \frac{n^2\alpha}{6}\left(J_1 - J_2\right) + \frac{n^2\alpha}{150}\left\{\frac{I_{31}}{\xi_1^2}\left(I_3 - I_{31}\frac{\xi^2}{\xi_1^2} - I_4\right) - \left(J_3 - J_{31}\frac{\xi^2}{\xi_1^2} - J_4\right)\right\}P_2 \qquad ...(5.1)$$

$$\sigma_{2e} = \sigma_{1e} + \frac{n^2\alpha}{6}\left(J_{11} - J_{21}\frac{\xi_1}{\xi}\right) + \frac{n^2\alpha}{150} I_{41}\left(\frac{J_{41}}{I_{41}} - \frac{I_{31}}{\xi_1^2}\right)\frac{\xi_1^3}{\xi^3} P_2 \qquad \ldots(5.2)$$

where

$$\left.\begin{aligned} J_1 &= \int_0^\xi y(I_1 - I_2)\,\theta^{n-1}\,dy;\; J_3 = \xi^2 \int_0^\xi \frac{(I_3 - I_4)}{y}\,\theta^{n-1}\,dy \\ J_2 &= \frac{1}{\xi}\int_0^\xi y^2(I_1 - I_2)\,\theta^{n-1}\,dy;\; J_4 = \frac{1}{\xi^3}\int_0^\xi y^4(I_3 - I_4)\,\theta^{n-1}\,dy. \end{aligned}\right\} \qquad \ldots(5.3)$$

One more iteration to the interior region gives satisfactory convergence and all integrals are evaluated as the Emden function is computed. The critical parameters, for each order of approximation, are shown in Table II.

Details of the exact values are taken from James (1964) for $n < 3$ and from Martin's (1970) second order solutions for $n \geqslant 3$. It will be seen that each sequence α_c, η_c alternates about the converged values, which are just Martin's first order results; this property is an aid in truncating the sequences. The convergence is more rapid for increasing polytropic index due to the higher exponent n entering the r.h.s. of (2.9). Also, as n increases the converged value, and indeed the third approximation, are closer to the exact values since neglected terms (which are essentially a measure of central condensation) are correspondingly smaller—c.f. Table I. Another advantage for large n is discussed in the next section.

The equatorial radius ξ_e is exhibited for moderate values of α in Tables III and IV for the cases $n = 2.5$ and $n = 3$ and compared with the corresponding results obtained by James. The agreement between the values obtained in the third approximation and James' results is somewhat spurious since the converged values will be close to, but greater, than the former. However, the error is less than 1% for all moderate α.

6. Surface Layers and General Discussion

In the foregoing procedure we have always neglected the mass beyond the Emden sphere and constructed a Roche envelope onto our interior solution. This proves to be a much better approximation than might at first be imagined in as far as extracting surface parameters is concerned—this is certainly borne out in a quantitative way from the results obtained by this means, especially when taken to second order in α (Martin 1970). We illustrate this by considering the special case $n = 2$.

Returning to (3.10) we expand the density function in orders of α thus :

Mid : $\sigma_1 = \sigma_{10}(\xi) + \alpha\sigma_{11}(\xi, \mu) + \alpha^2\sigma_{12}(\xi, \mu)$ for $\xi_1 \leqslant \xi \leqslant \xi_s$ ...(6.1)

Outer : $= \tau_{10}(\xi) + \alpha\tau_{11}(\xi, \mu) + \alpha^2\tau_{12}(\xi, \mu)$ for $\xi_s \leqslant \xi$ ...(6.2)

TABLE II

	α_c						$\eta_c = \xi_e/\xi_1$					
	Order of approximation				Conv. value	Exact value	Order of approximation				Conv. value	Exact value
	0	1	2	3			0	1	2	3		
1.5	3.29 (−2)	4.76 (−2)	4.07 (−2)	4.19 (−2)	4.16 (−2)	4.36 (−2)	1.5	1.418	1.453	1.450	1.451	1.467
2.0	1.73 (−2)	2.48 (−2)	2.05 (−2)	2.11 (−2)	2.10 (−2)	2.16 (−2)	1.5	1.411	1.451	1.445	1.446	1.445
2.5	8.21 (−3)	1.07 (−2)	9.58 (−2)	9.85 (−3)	9.80 (−3)	9.93 (−3)	1.5	1.423	1.457	1.449	1.451	1.450
3.0	3.63 (−3)	4.31 (−3)	3.98 (−3)	4.07 (−3)	4.06 (−3)	4.08 (−3)	1.5	1.439	1.467	1.459	1.461	1.459
3.5	1.29 (−3)	1.43 (−3)	1.37 (−3)	1.39 (−3)	1.39 (−3)	?	1.5	1.458	1.477	1.472	1.473	?
4.0	3.17 (−4)	3.35 (−4)	3.27 (−4)	3.29 (−4)	3.29 (−4)	3.29 (−4)	1.5	1.477	1.487	1.484	1.485	1.484

TABLE III

$n = 2.5 \quad (10^3\alpha_c = 9.93)$

$10^3\alpha$	ξ_e				
	Order of approximation				James' value
	0	1	2	3	
6.0	6.25	5.89	6.03	5.99	5.99
6.4	6.37	5.94	6.10	6.06	6.06
6.8	6.50	6.00	6.18	6.13	6.13
7.2	6.65	6.06	6.27	6.21	6.21
7.6	6.85	6.13	6.37	6.30	6.30
8.0	7.13	6.20	6.49	6.41	6.40
8.4	7.73	6.28	6.63	6.53	6.52
8.8	—	6.38	6.81	6.68	6.66
9.2	—	6.49	7.08	6.87	6.84

TABLE IV

$n = 3.0 \quad (10^3\alpha_c = 4.08)$

$10^3\alpha$	ξ_e				
	Order of approximation				James' value
	0	1	2	3	
0.5	7.05	7.01	7.02	7.02	7.02
1.0	7.22	7.13	7.17	7.16	7.16
1.5	7.42	7.27	7.34	7.32	7.32
2.0	7.67	7.44	7.53	7.51	7.51
2.5	7.98	7.63	7.78	7.74	7.74
3.0	8.43	7.88	8.10	8.03	8.03
3.5	9.31	8.21	8.58	8.46	8.46

It is important to recognise that these representations are 'exact' within the framework of the iteration. The σ's and τ's satisfy

$$\nabla^2\sigma_{10} = \xi_1^2\,\theta_1^2\left(1 - \frac{\xi_1}{\xi}\right)^2; \; \nabla^2\tau_{10} = 0 \qquad \text{...(6.3}a\text{)}$$

$$\nabla^2\sigma_{11} = \frac{\xi_1\theta_1}{6}\,\xi^2\left(1 - \frac{\xi_1}{\xi}\right)(1 - P_2) + 1; \; \nabla^2\tau_{11} = 1 \qquad \text{...(6.3}b\text{)}$$

$$\nabla^2\sigma_{12} = \frac{\xi^4}{36}\left(1 - P_2\right)^2; \nabla^2\tau_{12} = 0 \qquad \text{...(6.3c)}$$

and each is a linear combination of separated functions. The surface may be written approximately:

$$\xi_s = \xi_1 + \alpha\xi_{11}(\mu) + \alpha^2\xi_{12}(\mu). \qquad \text{...(6.4)}$$

Although the actual forms of ξ_{11} and ξ_{12} are irrelevant if, in the succeeding analysis, we retain only terms up to order α^2, it is assumed that this expression has been made as exactly as possible—e.g. least squares fit. Formally substituting the expansion (6.4) into (6.1) and (6.2) we obtain :

$$\left.\begin{aligned} \sigma_{10}(\xi_s) &= \sigma_{10}(\xi_1) + \alpha\sigma'_{10}(\xi_1)\,\xi_{11} + \alpha^2\left[\sigma'_{10}(\xi_1)\,\xi_{12} + \frac{\sigma''_{10}(\xi_1)\,\xi_{11}^2}{2}\right] + o(\alpha^3) \\ \alpha\sigma_{11}(\xi_s) &= \alpha\sigma_{11}(\xi_1, \mu) + \alpha^2\sigma'_{11}(\xi_1, \mu)\,\xi_{11} + o(\alpha^3) \\ \alpha^2\sigma_{12}(\xi_s) &= \alpha^2\sigma_{12}(\xi_1, \mu) + o(\alpha^3) \end{aligned}\right\} \qquad \text{...(6.5)}$$

and corresponding expansions for $\tau_{10}, \tau_{11}, \tau_{12}$. The primes denote partial differentiation w.r.t. ξ. Ensuring the continuity of σ_1 and its gradient across the surface gives, to zero order in α,

$$\sigma_{10}(\xi_1) = \tau_{10}(\xi_1); \; \sigma'_{10}(\xi_1) = \tau'_{10}(\xi_1). \qquad \text{...(6.6)}$$

From these relations and eqns. (6.3*a*) and their derived equations, we deduce the continuity of the second and third derivatives of σ_{10}. The first-order continuity conditions then become

$$\sigma_{11}(\xi_1, \mu) = \tau_{11}(\xi_1, \mu); \; \sigma'_{11}(\xi_1, \mu) = \tau'_{11}(\xi_1, \mu). \qquad \text{...(6.7)}$$

Equations (6.3*b*) then imply the continuity of the second derivatives, which give for the second-order conditions :

$$\sigma_{12}(\xi_1, \mu) = \tau_{12}(\xi_1, \mu); \; \sigma'_{12}(\xi_1, \mu) = \tau'_{12}(\xi_1, \mu) \qquad \text{...(6.8)}$$

Notice that each order (up to α^n for general integral n) is matched at the Emden sphere independent of the form of the matching surface. Higher orders explicitly involve the functions ξ_{11} and ξ_{12}. The net effect of matching the inner solution at $\xi = \xi_1$ with the *mid* solution, and then this with the outer solution at $\xi = \xi_s$ is to match the inner and outer solutions at the Emden sphere; this being just the method used in the previous sections. Notice that the tractability of the above argument depends on an accurate representation of σ_1 as a Taylor series about $\xi = \xi_1$ as given by eqns. (6.5). This is exact at the Emden sphere but becomes progressively worse as we approach ξ_s. On the other hand the neglect of mass in outer regions is a poor approximation at ξ_1 but improves as $\xi \rightarrow \xi_s$. As the matching procedure for both is identical we conclude that our iterative scheme gives a poor representation of the density in subsurface layers but elsewhere closer to the exact distribution than indicated by Table I. This is an important characteristic for rotating real stars since there we are chiefly concerned with an accurate description of surface characteristics, e.g. luminosity. For greater n we can retain integral powers of α up to order α^n thereby improving the Taylor series representation for σ_1. We can apply similar arguments to above to a hypothetical surface drawn interior to the Emden sphere. For example, the neglected terms at the surface given by

$$\theta(\xi) = \frac{\alpha\xi^2}{6}\left(1 - P_2\right)$$

would have magnitude $(2^n - (n+1))\,\theta^n$. The maximum value, ϵ_2, is shown in Table I.

The method described above is especially valuable for slow rotation and indeed our interest in rotating polytropes arose in connection with the solar rotation problem where a comprehensive discussion of the coupled mechanical and thermodynamical processes was necessary. This will be the subject of future communications.

Acknowledgement

This work was undertaken during the tenure of a National Research Council of Canada Post-Doctoral Fellowship and the author is indebted to Professor K. B. Ranger for making this support available. Numerical work was carried out using the IBM 360 system installed at the University of Toronto.

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