GRAVITATIONAL FIELDS OF THREE TYPES OF OBJECTS

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The field equations $R_{ij}=0$ give rise to three types of distinct Riemannian four-folds according as the 2-space $S^2(\xi)[d\xi^2+\sin^2\xi\, d\eta^2]$ in the space time metric

$$ds^2=-A(\rho)d\rho^2-\rho^2S^2(\xi)[d\xi^2+\sin^2\xi\, d\eta^2]+C(\rho)d\tau^2,$$

is of (i) constant positive curvature, (ii) constant negative curvature and (iii) zero curvature. It is shown that these three types of spaces correspond to the gravitational field of (a) ordinary material particle, (b) tachyon and (c) light-like particles respectively. An ordinary test mass at rest experiences repulsion in case of the last two types of fields. The case (a) leads to the usual Schwarzschild exterior field.

1. Introduction

In a recent article, Sudarshan (1968) has grouped physical objects into three classes: (I) which move at speeds less than that of light and can be observed at rest, (II) which move with the speed of light and cannot be made to travel any slower or any faster and (III) which travel at a speed greater than the speed of light. The name tachyon is given to particles which move faster than light. It is known that the existence of a free tachyon is not inconsistent with the consequences of the basic principles of relativity (Camenzind 1970).

The gravitational field of class (I) objects is described by the well-known Schwarzschild exterior metric. In this paper it is shown that under the framework of general relativity, Schwarzschild-like gravitational fields exist for objects of class (II) and class (III) also. The geometrical and physical significance of such fields is discussed.

2. The Line Element and the Solution of the Field Equations

We consider the line-element

$$ds^2=-Ad\rho^2-\rho^2S^2\{d\xi^2+\sin^2\xi\, d\eta^2\}+C\, d\tau^2 \qquad \ldots(2.1)$$

where

$$A=A(\rho), \qquad C=C(\rho), \qquad S=S(\xi). \qquad \ldots(2.2)$$

In consequence of (2.1) and (2.2), the field equation $R_{ij}=0$ lead to the following relations:

$$C_{11}-\frac{A_1C_1}{2A}-\frac{C_1^2}{2C}+\frac{2C_1}{\rho}=0 \qquad \text{... (2.3)}$$

$$\frac{A_1}{A}+\frac{C_1}{C}=0 \qquad \text{... (2.4)}$$

$$\frac{A_1}{2A}-\frac{C_1}{2C}-\frac{1}{\rho}=\frac{KA}{\rho} \qquad \text{... (2.5)}$$

$$SS_{22}-S_2^2+SS_2\cot\theta=KS^4+S^2. \qquad \text{... (2.6)}$$

Here the subscripts 1 and 2 denote a partial differentiation with respect to ρ and ξ respectively, and K is a real constant. We consider the three cases corresponding to (i) $K<0$, (ii) $K>0$ and (iii) $K=0$.

Case I—When $K<0$, let $K=-k^2$, eqn. (2.6) admits the general solution

$$S(\xi)=\frac{2\alpha}{K\sin\xi}\quad\frac{\left(\beta\tan\frac{\xi}{2}\right)^{\alpha}}{\left\{1+\left(\beta\tan\frac{\xi}{2}\right)^{-\alpha}\right\}} \qquad \text{... (2.7)}$$

where α and β are arbitrary constants.

Equations (2.3)-(2.5) admit the solutions

$$A=\left(k^2+\frac{k_1}{\rho}\right)^{-1},\qquad C=k^2+\frac{k_1}{\rho} \qquad \text{... (2.8)}$$

where k_1 is an arbitrary constant.

Case II—When $K>0$, let $K=k^2$. Equation (2.6), then, admits the general solution

$$S(\xi)=\frac{2a}{k\sin\xi}\quad\frac{\left(b\tan\frac{\xi}{2}\right)^{a}}{\left\{1-\left(b\tan\frac{\xi}{2}\right)^{2a}\right\}} \qquad \text{... (2.9)}$$

where a and b are constants of integration. In this case eqns. (2.3)–(2.5) imply that

$$A=\left(-k^2+\frac{k_2}{\rho}\right)^{-1},\qquad C=-k^2+\frac{k^2}{\rho} \qquad \text{...(2.10)}$$

where k_2 is an arbitrary constant.

Case III—When $K=0$, eqn. (2.6) gives

$$S(\xi)=\frac{\left(\tan \frac{\xi}{2}\right)^{c_1}}{c_2 \sin \xi} \qquad \text{...(2.11)}$$

where c_1 and c_2 are arbitrary constants. In this case the functions A and C, satisfying (2.3)-(2.5), assume the form

$$A=\left(\frac{k_3}{\rho}\right)^{-1}, \quad C=\frac{k_3}{\rho} \qquad \text{...(2.12)}$$

where k_3 is a constant.

3. The Geometrical and Gravitation Significance of the Solutions

In case I, the 2-space

$$S^2(\xi)\ (d\xi^2+\sin^2 \xi d\,\eta^2) \qquad \text{...(2.13)}$$

reduces to

$$d\theta^2+\alpha^2 \sin^2 \theta\, d\varphi^2 \qquad \text{...(2.14)}$$

by means of the coordinate transformation

$$\theta=2 \tan^{-1} \left(\beta \tan \frac{\xi}{2}\right)^{\alpha}. \qquad \text{...(2.15)}$$

The line-element (2.14) is isometric with the geometry on a sphere. According as α is greater than, equal to, or less than unity, one gets three types of surfaces of revolution of constant positive curvature (Goetz 1970).

When $\alpha = 1$, (2.7) can be expressed as

$$S^2(\xi) = \frac{1-\nu^2}{k^2}(1-\nu \cos \xi)^{-2} \qquad \text{...(2.16)}$$

where ν is a constant given by $\beta^2 = \frac{1+\nu}{1-\nu}$, $\nu<1$.

If we set $k_1 = -2m$ and $k^2 = 1-\nu^2$ in (2.8), then, as a consequence of (2.11), the line-element (2.1), can be expressed as

$$ds^2 = -\left(1-\nu^2-\frac{2m}{\rho}\right)^{-1} d\rho^2 - \rho^2 (1-\nu \cos \xi)^{-2}\, d\sigma^2 + \left(1-\nu^2-\frac{2m}{\rho}\right)d\tau^2 \qquad \text{...(2.17)}$$

where $d\sigma^2 = d\xi^2 + \sin^2 \xi \, d\eta^2$ and v is a constant which is less than one. When $v = 0$, (2.17) reduces to the usual Schwarzschild exterior metric. If we use the transformation equations

$$\left.\begin{aligned} \sin \theta &= \frac{\sqrt{1 - v^2} \sin \xi}{1 - v \cos \xi} \\ \rho &= \sqrt{1 - v^2}\, r, \quad t = \sqrt{1 - v^2}\, \tau \\ M &= m(1 - v^2)^{-3/2}, \quad \eta = \varphi \end{aligned}\right\} \qquad \ldots(2.18)$$

in (2.17), the usual Schwarzschild metric is recovered. The first relation in (2.13) is exactly the relation between the two sets of angles in the aberration equation derived from Lorentz transformation. Hence the parameter v can be interpreted as the velocity of the gravitating object relative to a fixed origin in the direction $\theta = 0 = \xi$, against the flat background.

Consider the Case II. In view of (2.9), the two-space $S^2(d\xi^2 + \sin^2 \xi \, d\eta^2)$ goes into the form $d\theta^2 + a^2 \sinh^2 \theta \, d\varphi^2$ by means of transformation

$$\theta = \log \frac{1 + \left(b \tan \frac{\xi}{2}\right)^a}{1 - \left(b \tan \frac{\xi}{2}\right)^a}, \qquad \ldots(2.19)$$

$$\varphi = \eta.$$

When $a = 1$, the two-space is a pseudo-spherical surface of constant (negative) curvature. When $a \neq 1$, one gets the two distinct types of surfaces of constant negative curvature corresponding to $a > 1$ or $a < 1$.

If we set $a = 1$ and $b^2 = \frac{v + 1}{v - 1}$, $v > 1$ in (2.9), then we find that

$$S^2(\xi) = \frac{v^2 - 1}{k^2} (1 - v \cos \xi)^{-2} \qquad \ldots(2.20)$$

In consequence of (2.20), (2.10) and (2.1), we obtain the line-element

$$ds^2 = -\left(1 - v^2 + \frac{2m}{\rho}\right)^{-1} d\rho^2 - \rho^2(1 - v \cos \xi)^{-2}(d\xi^2 + \sin^2 \xi \, d\eta^2)$$
$$+\left(1 - v^2 + \frac{2m}{\rho}\right) d\tau^2 \qquad \ldots(2.21)$$

where k^2 in (2.20) and (2.10) is equated to $v^2 - 1$ and $k_2 = 2m$, a positive constant.

We recall here that ν is greater than one in this case. By the use of the transformation equations

$$\left.\begin{aligned} \sinh\theta &= \frac{\sqrt{\nu^2-1}\sin\xi}{1-\nu\cos\xi} \\ \rho = \sqrt{\nu^2-1}\, r,\ & t = \sqrt{\nu^2-1}\ \tau \\ \eta = \varphi,\quad \nu>1,\quad & M = m(\nu^2-1)^{-3/2} \end{aligned}\right\} \qquad ...(2.22)$$

the metric (2.21) goes into the form

$$ds^2 = -\left(\frac{2M}{r}-1\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sinh^2\theta d\varphi^2) + \left(\frac{2M}{r}-1\right)dt^2 \qquad ...(2.23)$$

where $0 < r < 2M$, $0 \leqslant \theta < \infty$.

Vaidya (1971) has described the metric (2.21) as the gravitational field of a tachyon, an object moving with a speed greater than that of light. By considering the geodesic equations, it can be shown that an ordinary test particle at rest will experience repulsion in the field (2.23).

In case III, from (2.1), (2.11), (2.12) with $k_3 = 2m$, $m > 0$, we have the line-element

$$ds^2 = -\frac{d\rho^2}{(2m/\rho)} - \rho^2(d\theta^2 + \theta^2 d\varphi^2) + \frac{2m}{\rho}d\tau^2 \qquad ...(2.24)$$

where we have set $c_1 = 1$, $c_2 = 1$ and $\xi = 2\tan^{-1}\theta$, in (2.11).

The metric (2.24) describes a gravitational field satisfying $R_{ij} = 0$, which is distinct from Case I and Case II discussed above. As compared to (2.21), this may be looked upon as the gravitational field of an object moving with the fundamental speed, that is, the speed of light. By a simple coordinate transformation the metric (2.24) can be thrown into the form

$$ds^2 = -\,dx^2 - \left(\frac{3\sqrt{2m}}{2}x\right)^{4/3} (dy^2 + dz^2 + 2m\left(\frac{3\sqrt{2m}}{2}\right)^{-2/3} dT^2 \qquad ...(2.25)$$

which is plane-symmetric in the sense of Taub (1951). Geodesic equations reveal that an ordinary test-mass at rest in the field (2.25) will experience repulsion.

4. Conclusion

The field equations $R_{ij} = 0$ give rise to three types of distinct Riemannian four-folds according as the 2-space $S^2(\,d\xi^2 + \sin^2\xi\; d\eta^2)$ in (2.1) is of (i) constant positive curvature, (ii) constant negative curvature and (iii) zero curvature. When this 2-space is of constant positive curvature, the resulting gravitational field is identified with the field due to an object moving with a speed less than that of light. In case of constant negative curvature, the field of a tachyon, according to Vaidya, has been obtained. When the curvature is zero, the resulting gravitational field is interpreted as the field due to an object moving with the speed of light.

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