TOTAL MANIFOLDS AND REFLEXIVITY IN NORMED LINEAR SPACES†

by LAL BAHADUR*, *S. N. Govt. P. G. College, Khandwa (M.P.)*

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Ruston (1957) has proved that a Banach space is reflexive if and only if B^* contains no proper closed total linear manifolds. In the present paper it is generalized that a normed linear space N is reflexive $\Leftrightarrow$ N^* contains no total zero-hyperplane $\Leftrightarrow$ N^* contains no proper closed total subspaces. The characterizations of (i) total zero-hyperplanes and (ii) reflexive normed space N as a space such that each total subspace of N^* is (norm) dense in N^* are also given.

In what follows, N and N^* stand for a normed linear space and its dual respectively. Similarly, B and B^* stand for a Banach space and its dual respectively.

Ruston (1957) has proved that a Banach space B is reflexive if and only if B^* contains no proper closed total linear manifolds. Theorem 2 of the present paper generalizes the above result by proving that a normed linear space N is reflexive if and only if N^* does not contain any total zero-hyperplane. Theorem 3 is another generalization of the same result as it proves that a normed space N is reflexive if and only if N^* does not contain any proper closed total subspace. Theorems 2 and 3 of this paper also generalise Theorem 3.1 of Wilder (1967).

A characterization of total zero-hyperplanes in N^* has been given in Theorem 1. Lastly Theorem 4 characterizes reflexive normed space as a space such that each total subspace in N^* is (norm) dense in N^*.

Definition 1 (Total manifold)—Let N be a normed linear space. A subspace S of N^* is said to be a total manifold if $f(x) = 0$ for all f in S implies $x = 0$. By Hahn-Banach theorem, it is easy to derive that N^* itself is a total manifold. Further, it is obvious that if S is total and $T \supset S$ then T is also total.

Definition 2 (A zero-hyperplane or a hyperplane through zero)—Let $f \in N^*$, $(f \neq 0)$, then the kernel of f (i.e. the set of all those points of N where f is zero) is said to be a zero-hyperplane or a hyperplane through zero. If M is a zero-hyperplane and $x \in N$ then $x + M$ is said to be a hyperplane in N. Clearly, every hyperplane in N is (norm) closed.

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*Present address : Govt. College, Sehore (M.P.), Pin 466001.

The following theorem characterizes total zero-hyperplanes of N^*.

Theorem 1—A zero-hyperplane M of N^* is or is not total according as its functional, say F (which lies in N^{**}) does not or does belong to $Q(N)$, where Q is the natural embedding of N into N^{**}.

Or

A zero-hyperplane of N^* is total if and only if its functional F (hence $F \neq 0$) does belong to $Q(N)$, where Q is the natural embedding of N into N^{**}.

PROOF : For any $x \in N$ we will write F_x for $Q(x)$. As $F \in Q(N)$, $F = F_x$ (for some $x \in N$, $x \neq 0$)

$\Rightarrow F_x(f) = 0, \forall f \in M.$
$\Rightarrow f(x) = 0, \forall f \in M$, though $x \neq 0$
$\Rightarrow M$ is not total.

Conversely, M is not total implies that there exists $x \in N$, $x \neq 0$ such that $f(x) = 0, \forall f \in M$

$\Rightarrow F_x(f) = 0, \forall f \in M$
$\Rightarrow F = a F_x$, for some scalar a
$\Rightarrow F = F_{ax}$, by property of natural embedding
$\Rightarrow F \in Q(N)$.

Ruston (1957) has proved that a Banach space B is reflexive if and only if B^* contains no proper closed total linear manifolds Similar result was proved in Theorem 3.1 of Wilder (1967). The following theorem is a more complete result.

Theorem 2—A normed linear space N is reflexive if and only if N^* does not contain any total zero-hyperplanes.

PROOF : N is reflexive $\Rightarrow$ there is no functional F in N^{**} which does not belong to $Q(N) \Rightarrow$ no zero-hyperplane in N^* is total, by Theorem 1. Conversely, N^* contains no total zero-hyperplane $\Rightarrow$ no functional of N^{**} is out of $Q(N)$, from Theorem 1, $\Rightarrow N^{**} = Q(N) \Rightarrow N$ is reflexive.

Corollary 1—If N^* contains no total zero-hyperplanes then N is a Banach space.

PROOF : N^* contains no total zero-hyperplane

$\Rightarrow N$ is reflexive (from Theorem 2)
$\Rightarrow N$ is complete (as N^{**} is complete and N and N^{**} are isometrically isomorphic).

Corollary 2—N is reflexive $\Rightarrow$ there cannot be a total subspace M of N^*, for which there exists an element $f_0 \in N^*$ such that $d(f_0, M) = d > 0$.

Proof : Suppose, on the contrary that there exists a total subspace M and $f_0 \in N^*$ such that, $d(f_0, M) = d > 0$, then by Corollary 2 of Lusternik and Sobolev (1967, p. 167) there exists $F \in N^{**}$ such that, $F(f_0) = 1$ and $F(f) = 0$, for all f in M. If $H = \ker F$, then $H \supset M$. As M is total, H is also total. Now H is a zero-hyperplane in $N^* \Rightarrow N$ is not reflexive, by Theorem 2.

Corollary 3—N is reflexive $\Rightarrow N$ cannot contain any proper closed total subspace.

The proof is easy in view of Corollary 2 and is, therefore, omitted.

The following theorem generalizes the result of Ruston (1957) and Wilder (1967) from Banach space to any normed linear space.

Theorem 3—A normed space N is reflexive if and only if N^* contains no proper closed total subspaces.

Proof : N^* contains no proper closed total subspaces

$\Rightarrow N^*$ contains no total zero-hyperplanes.
$\Rightarrow N$ is reflexive (by Theorem 2).

The other way it follows from Corollary 3.

Corollary 4—A normed space N is reflexive $\Rightarrow$ each total subspace M in N^* is dense (in N^*).

Theorem 4—A normed space N is reflexive if and only if each total subspace M in N^* is dense in N^*.

Proof : Each total subspace M is dense in $N^* \Rightarrow$ any total closed subspace in N^* cannot be a proper subspace

$\Rightarrow N$ is reflexive, from Theorem 3.

The other way it follows from Corollary 4.

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