BANACH SPACE OF VECTOR-VALUED LIPSCHITZ FUNCTION

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Leeuw (1961–62) has proved that the Banach space $C^a([0, 1], C)$ of complex-valued Lipschitz functions on [0, 1], is isometrically isomorphic with a conjugate space of a Banach space if $0<a<1$. The same result has been proved for $a=1$ by Roy (1966). The theorem given in this paper generalizes these results. In this theorem we prove that if X is a compact metric space and B is a Banach space, then the Banach space — $C^a(X, B)$ where $0<a\leqslant 1$, of all Lipschitz functions from X into B is isometrically isomorphic with a Banach space $L(S, B)$.

Let X be a compact metric space with the metric d and B be a Banach space. A function $f : X \to B$ is said to be a Lipschitz function (for a fixed a, $0 < a \leqslant 1$) if,

$$\sup_{\substack{x, y \in X \\ x \neq y}} \|f(x)-f(y)\| d(x, y)^{-a}$$

which is denoted by $\|f\|_a$, is finite. All Lipschitz functions for a fixed a from X into B form a normed linear space with the norm,

$$\|f\| = \text{Max}\,(\|f\|_\infty, \|f\|_a)$$

where

$$\|f\|_\infty = \max_{x \in X} \|f(x)\|.$$

This normed linear space is denoted by $C^a(X, B)$. The fact that $\| \ \|$ is a norm has been proved by Lal Bahadur (1971).

If N and N' are normed linear spaces, then the normed linear space of all linear continuous functions from N into N' (with the usual norm) is denoted by $L(N, N')$. In what follows, N^* will denote the conjugate space of a normed linear space N.

The following proposition has been proved by Lal Bahadur (1971) and is being given here for the sake of completeness.

Proposition—$C^a(X, B)$ is a Banach space.

PROOF : Let $\{f_n\}$ be a Cauchy sequence in $C^a(X, B)$. Then given $\varepsilon>0$, there

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exists N such that,

$$\|f_n - f_m\| < \varepsilon, \text{ if } n > N \text{ and } m > N.$$

Hence we have,

$$\|f_n - f_m\|_\infty < \varepsilon, \text{ if } n > N \text{ and } m > N \qquad \text{...(1)}$$

and

$$\|f_n - f_m\|_a < \varepsilon, \text{ if } n > N \text{ and } m > N. \qquad \text{.. (2)}$$

From (1), it follows that $\{f_n(x)\}$ is a Cauchy sequence in B for each x in X. As B is complete, it converges to an element of B which is denoted by $f(x)$. Thus f is a function from X into B.

As a Cauchy sequence is bounded we have a constant K such that,

$\|f_n\| < K$, for all n. Now,

$$\|f(x) - f(y)\| = \lim_{n \to \infty} \|f_n(x) - f_n(y)\| \leqslant \operatorname{Sup}_n \|f_n\|_a \, d(x, y)^a \leqslant K \, d(x, y)^a$$

Hence $f \in C^a(X, B)$

Again from (1), $\| f_n(x) - f_m(x) \| < \varepsilon$, for all x in X, if $m > N$ and $n > N$. Now as the sequence (for fixed $n > N$),

$$f_n(x) - f_1(x), f_n(x) - f_2(x), \quad \ldots\ldots, \; f_n(x) - f_m(x), \ldots\ldots$$

converges to $f_n(x) - f(x)$, we must have,

$$\| f_n(x) - f(x) \| \leqslant \varepsilon, \text{ for all } x \in X.$$

Hence

$$\| f_n - f \|_\infty \leqslant \varepsilon, \text{ for all } n > N. \qquad \text{...(3)}$$

Again from (2),

$\|(f_n - f_m)(x) - (f_n - f_m)(y)\| \leqslant \varepsilon d(x, y)^a$, if $m > N$ and $n > N$, for all x and y in X. Or $\|(f_n(x) - f_n(y)) - (f_m(x) - f_m(y))\| \leqslant \varepsilon d(x, y)^a$. Now for a fixed $n > N$, the sequence $\{f_n(x) - f_n(y) - (f_m(x) - f_m(y))\}$ converges to $f_n(x) - f_n(y) - (f(x) - f(y))$.

Hence

$\|f_n(x) - f_n(y) - (f(x) - f(y))\| \leqslant \varepsilon d(x, y)^a$, for all x, y in X if $n > N$. Or
$\|(f_n - f)(x) - (f_n - f)(y)\| \leqslant \varepsilon d(x, y)^a$.

Hence

$$\| f_n - f \|_a < \varepsilon. \text{ if } n > N. \qquad \text{...(4)}$$

Combining (3) and (4) we get,

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Hence $C^a(X, B)$ is complete.

For $x \in X$, let T_x denote the map on $C^a(X, B)$ given by

$T_x(f) = f(x)$ for all $f \in C^a(X, B)$.

Then T_x is a linear map from $C^a(X, B)$ into B. Now

$$\|T_x(f)\| = \|f(x)\| \leqslant \|f\|_\infty \leqslant \|f\|$$

implies that T_x is continuous and

$$\|T_x\| \leqslant 1. \qquad \text{...(5)}$$

Also $\|(T_x - T_y)(f)\| = \|f(x) - f(y)\| \leqslant \|f\|_a\, d(x, y)^a \leqslant \|f\|\, d(x, y)^a$

implies $\|T_x - T_y\| \leqslant d(x, y)^b$. ...(6)

Let Y denote the space $C^a(X, B)$. The closure of linear space of $\{T_x\}_{x \in X}$ in $L(Y, B)$ is denoted by S. Thus S itself is a Banach space. Now,

Theorem—$C^a(X, B)$, $0 < a \leqslant 1$, is isometrically isomorphic with $L(S, B)$.

PROOF : We write Y for $C^a(X, B)$.

For F in $L(S, B)$ define a map $\overline{F}$ on X by $\overline{F}(x) = F(T_x)$.

Now $\|\overline{F}(x)\| = \|F(T_x)\| \leqslant \|F\|\,\|T_x\| \leqslant \|F\|$, from (5).

Hence $\|\overline{F}\|_\infty \leqslant \|F\|$.

Also $\|\overline{F}(x) - \overline{F}(y)\| = \|F(T_x) - F(T_y)\| = \|F(T_x - T_y)\|$

$\leqslant \|F\|\,\|T_x - T_y\|$

$\leqslant \|F\|\, d(x, y)^a$, from (6).

Hence $\|\overline{F}\|_a \leqslant \|F\|$.

Thus $\overline{F} \in Y$ and $\|\overline{F}\| \leqslant \|F\|$. ...(7)

It is easy to see that the map $Q : L(S, B) \to Y$, $F \to \overline{F}$, is linear. We show that Q is onto and norm-preserving. For f in y, define F_f on S by $F_f(T) = T(f)$, for all T in S. Now F_f in linear, and $\|F_f(T)\| = \|T(f)\| \leqslant \|T\|\,\|f\|$, implies that F_f is continuous and thus $F_f \in L(S, B)$. Also we have

$$\|F_f\| \leqslant \|f\| \qquad \text{.. (8)}$$

Now $\overline{F_f}(x) = F_f(T_x) = T_x(f) = f(x)$, implies that $\overline{F_f} = f$. Hence the map Q is onto.

Now writing F for F_f and writing $\overline{F}$ for f in (8) we have,

$$\| F \| \leqslant \| \overline{F} \| \qquad \ldots(9)$$

Now (7) and (9) imply that $\| F \| = \| \overline{F} \|$. Thus Y and $L(S, B)$ are isometrically isomorphic.

Corollary—Writing C, the Banach space of all complex numbers, in place of B and $X = [0, 1]$ in above theorem, it follows that $C^a([0, 1], C)$ is isometrically isomorphic with S^* (the conjugate space of the Banach space S) for $0 < a < 1$.

This result has been proved by Leeuw (1961–62) for $0 < a < 1$ and for $a = 1$ it has been proved by Roy (1966).

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