TORSIONAL VIBRATIONS OF A PRE-TWISTED CANTILEVER BEAM

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The differential equation for the torsional vibrations of a pre-twisted cantilever beam of rectangular cross-section has been obtained. The Galerkin's method is used to obtain the frequencies of various modes of vibration. It has been found that the frequencies of various modes may be considerably increased depending upon the thinness of the bar and the amount of pre-twist.

INTRODUCTION

The analysis presented in this paper considers torsional vibrations of a pre-twisted cantilever beam of rectangular cross-section. It is an attempt to discuss torsional vibrations of a pre-twisted beam, as the torsional vibration problems have received less attention so far as compared to lateral vibrations. It is well known that when a thin bar is under torsion there is a slight decrease in distance between cross-sections. A normal stress is thus produced in each longitudinal fibre which is not parallel to the axis of bar, and hence there is a stress component which produces an additional torque.

THE DIFFERENTIAL EQUATIONS

The differential equation for the bar of length l, pre-twisted by an amount τ and having axis of x through the centroids of cross-sectional planes will be obtained by considering the additional stresses produced in the longitudinal fibres.

Considering an element of length dx of the bar and a longitudinal fibre at a distance r from the axis of the bar, Diprima (1959) obtained strain in the fibre due to the rotation θ as

$$\varepsilon = r^2 \beta \frac{\partial \theta}{\partial x} \qquad \ldots(1)$$

*Present address: Lecturer in Mathematics, Regional College of Education, NCERT, Shyamla Hill, Bhopal 462013 where β is the twist in radians per unit length of x-axis and θ is the angle through which each cross-section rotates and is measured in the same manner as τ.

The normal stress $\sigma = E\varepsilon$ has components

$$\sigma \cos r\beta \simeq \sigma . 1 \quad \ldots(2)$$

and

$$\sigma \sin r\beta \simeq \sigma r\beta \quad \ldots(3)$$

parallel and perpendicular to the axis of bar and E is the Young's modulus of elasticity.

Since the state of deformation is not uniform the relation between the variable torque M_t and the angle of twist θ is given by (Timoshenko 1955).

$$M_t = GJ\frac{\partial \theta}{\partial x} - C_1 \frac{\partial^3 \theta}{\partial x^3} \quad \ldots(4)$$

where GJ is the torsional rigidity for uniform torsion and C_1 is the warping rigidity.

On considering the torque due to pre-twist the resultant torque T, acting across a cross-section is given by

$$T = GJ\frac{\partial \theta}{\partial x} - C_1 \frac{\partial^3 \theta}{\partial x^3} + \int_A r(r\beta\sigma)\, dA. \quad \ldots(5)$$

The term $\int_A r(r\beta\sigma)\, dA$, has occurred due to torque formed by the perpendicular force given by eqn. (3), where A is the area of cross-section.

Substituting $\sigma = E\varepsilon$ in eqn. (5) and using eqn. (1), on simplification one gets

$$T = (GJ + E\beta^2 I_4)\frac{\partial \theta}{\partial x} - C_1 \frac{\partial^3 \theta}{\partial x^3} \quad \ldots(6)$$

where

$$I_4 = \int_A r^4\, dA$$

Thus the equation of motion for an element dx is

$$- T + T + \frac{\partial T}{\partial x} dx = \rho I_2 \frac{\partial^2 \theta}{\partial t^2}\, dx \quad \ldots(7)$$

where

$I_2 = \int_A r^2\, dA$, and ρ is the density of the material.

Simplifying and substitution for T from eqn. (6), one may get the governing differential equation as

$$(GJ + EI_4\beta^2)\frac{\partial^2\theta}{\partial x^2} - C_1\frac{\partial^4\theta}{\partial x^4} = \rho I_2\frac{\partial^2\theta}{\partial t^2}\,. \qquad \ldots(8)$$

This equation is similar to the one obtained by Carnegie (1962) using energy principle.

Determination of Natural Frequencies

Equation (8) is to be put now in terms of a dimensionless variable $\xi = \left(\frac{x}{l}\right)$ and the substitutions

$$b_1 = \frac{GJ + EI_4\beta^2}{\rho I_2 l^2} \text{ and } b_2 = \frac{C_1}{\rho I_2 l^4}$$

are used to get the equation

$$b_1\frac{\partial^2\theta}{\partial\xi^2} - b_2\frac{\partial^4\theta}{\partial\xi^4} = \frac{\partial^2\theta}{\partial t^2}\,. \qquad \ldots(9)$$

The solution of eqn. (9) is of the form

$$\theta(\xi, t) = \Theta(\xi)e^{i\omega t} \qquad \ldots(10)$$

where $\Theta(\xi)$ is a function of ξ only.

The function $\Theta(\xi)$ satisfies all the boundary conditions of the beam which are as follows:

$$\left.\begin{aligned} \Theta(\xi) = \frac{d^2\Theta}{d\xi^2} &= 0;\ at\ \xi = 0 \\ \frac{d\Theta}{d\xi} = \frac{d^3\Theta}{d\xi^3} &= 0;\ at\ \xi = 1. \end{aligned}\right\} \qquad \ldots(11)$$

Further, the substitution of solution (10) in eqn. (9), yields the equation

$$b_1\frac{d^2\Theta}{d\xi^2} - b_2\frac{d^4\Theta}{d\xi^4} + \omega^2\Theta = 0. \qquad (12)$$

For the determination of frequencies for various modes of torsional vibrations of the pre-twisted beam, $\Theta(\xi)$ is chosen as

$$\Theta(\xi) = A_r \sin\left(\frac{\overline{2r+1}\,\pi}{2}\,\xi\right);\ r = 0, 1, 2, \ldots \qquad \ldots(13)$$

$$= A_r f_r \text{ (say)};\ r = 0, 1, 2, \ldots$$

Substitution of the solution given by (13) in (12) and performing the required differentiations, the discrepancy or the error $\varepsilon_r(\xi)$ in the differential equation (12) can be shown to be

$$\varepsilon_r(\xi) = A_r\,(b_1 f_r'' - b_2 f_r'''' + \omega^2 f_r)\ ;\ r = 0, 1, 2, \ldots \qquad \ldots(14)$$

According to Galerkin's process

$$\int_0^1 \varepsilon_r(\xi)\, f_r\, d\xi = 0,\ r = 0, 1, 2, 3, \ldots \qquad \ldots(15)$$

Substitution of (13) and (14) in (15) and making necessary operations one obtains

$$\omega^2 = \left[\frac{(2r+1)\,\pi}{2}\right]^2 \cdot \left[b_1 + \left\langle \frac{2r+1.\,\pi}{2} \right\rangle^2 b_2\right], \qquad \ldots(16)$$
$$r = 0, 1, 2, 3, \ldots$$

The torsional frequencies thus may be obtained for various modes by putting respectively $r = 0, 1, 2, 3, \ldots\ldots$ in eqn. (16).

Numerical Example

A numerical example for the torsional vibrations of a pre-twisted beam is now presented to study (i) the effect of pre-twist on the frequencies of first three modes of vibrations and (ii) the effect of the thinness of the pre-twisted beam on the fundamental frequency of vibration.

The frequencies are computed from eqn. (16) and the cross-section of the beam is taken as rectangular having width $2b$ and depth $2h$. The formulae

$$J = 4K_1 A h^2$$

and

$$C_1 = \frac{4}{3}\,(E\, h^3\, b^3) \qquad \ldots(17)$$

are used to calculate J and C_1, also table for K_1 is as follows (Timoshenko and Goodier 1951).

$\frac{h}{b}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
K_1	0.229	0 263	0.281	0.291	0.312

The other physical constants of the beam are taken as follows:

$$E = 30 \times 10^6 \text{ lb/in}^2, \quad G = 11.25 \times 10^6 \text{ lb/in}^2, \quad l = 8 \text{ in}, \quad b = 1 \text{ in},$$

and $\rho = 0.00082 \text{ lb/in}^3$.

With these values, the frequencies for the first three modes of torsional vibrations for five different values of β and for $h/b = 1/5$ have been computed and tabulated as below:

	Torsional frequencies (ω)		
	No. of modes		
β	Fundamental mode	First mode	Second mode
0.0	4.36231×10^4	14.50805×10^4	28.32915×10^4
0.1	4.36270×10^4	14.50910×10^4	28.33065×10^4
0.2	4.36387×10^4	14.51226×10^4	28.33513×10^4
0.3	4.36580×10^4	14.51751×10^4	28.34261×10^4
0.4	4.36852×10^4	14.52486×10^4	28.35306×10^4

Further, the fundamental frequencies for three different values of h/b and five different values of β have also been computed and tabulated as below:

	Fundamental frequencies (ω)		
	h/b		
β	1/3	1/4	1/5
0.0	4.15345×10^4	4.28889×10^4	4.36231×10^4
0.1	4.15458×10^4	4.28951×10^4	4.36270×10^4
0.2	4.15798×10^4	4.29136×10^4	4.36387×10^4
0.3	4.16363×10^4	4.29444×10^4	4.36581×10^4
0.4	4.17154×10^4	4.29875×10^4	4.36852×10^4

In Figs. 1 and 2, $\lambda = \omega/\omega_{\beta=0}$, i.e. the ratio of the frequency of the pre-twisted beam to the frequency of the beam when the pre-twist is zero, has been plotted as a function of β.

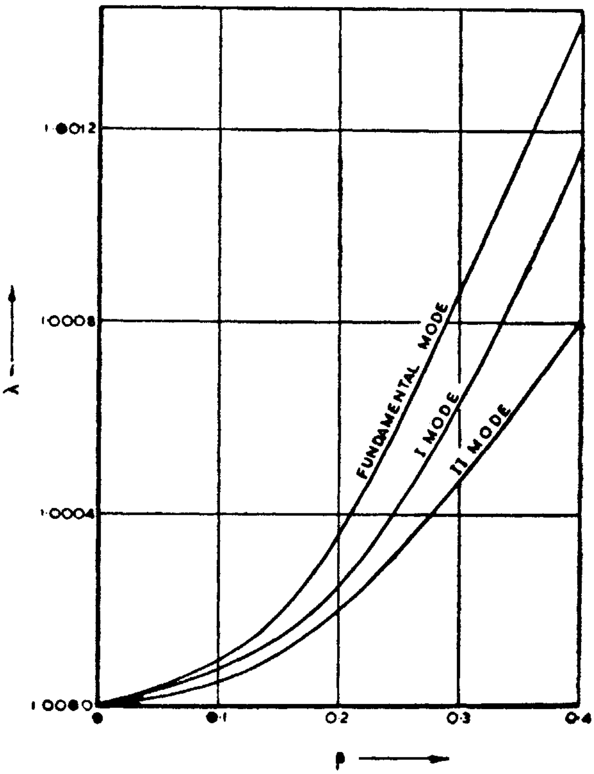

FIG. 1. Effect of pre-twist on torsional frequencies of first three modes.

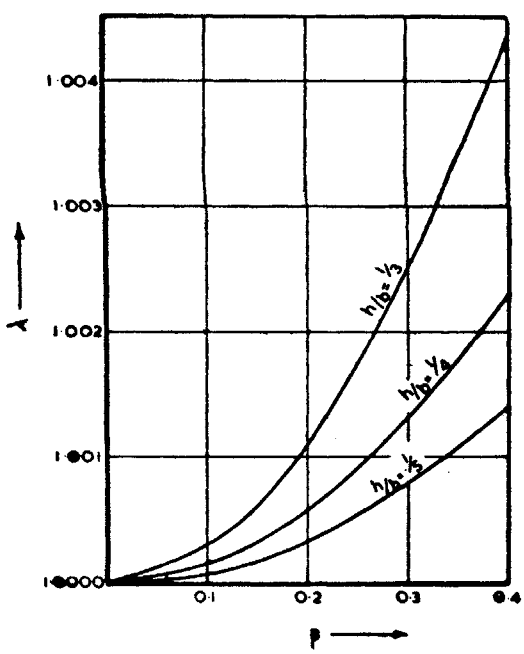

FIG. 2. Effect of thinness beam on fundamental frequency for various values of pre-twists.

Conclusion

The Galerkin's method used here to solve the equation of motion gives an upper bound of the frequencies. The results obtained for various frequencies of torsional vibrations show that it increases with the amount of pre-twist and the thinness of the beam.

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