ON H-PROJECTIVELY FLAT KÄHLER SUBMANIFOLD IN A KÄHLER MANIFOLD

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(*Received* 24 *November* 1972)

Mishra (1972) has obtained the conditions that an almost Hermite submanifold V_{2m-2} in a Kähler manifold V_{2m} is Kähler and a few relations are also found in the second fundamental tensors H and K. In this paper, the authors have obtained some more relations in H and K and used these results to find out the consequences for the flatness of H-projective curvature tensor in V_{2m-2} and V_{2m}.

1. INTRODUCTION

Let V_{2m} be a $2m$ dimensional almost Hermite manifold with the structure (F, G) satisfying

$$F(F(\lambda)) = -\lambda \quad \text{..(1.1)}$$

$$G(F(\lambda),\ F(\mu)) = G(\lambda, \mu) \quad \text{...(1.2)}$$

for arbitrary vectors λ, μ.

Agreement 1.1—In this paper the equations containing X, Y, Z, U will hold for arbitrary vector fields X, Y, Z, U in V_{2m-2}.

Let us put

$$'F(\lambda, \mu) \overset{def}{=} G(F(\lambda), \mu), \text{ then } 'F \text{ is skew-symmetric.} \quad (1.3)$$

An almost Hermite manifold V_{2m} for which

$$(E_\lambda F)(\mu) = 0 \quad \text{...(1.4)}$$

(where E is the Riemannian connexion in V_{2m}, is satisfied), is called a Kähler manifold.

Let V_{2m-2} be a submanifold of V_{2m} with b

$$b: V_{2m-2} \longrightarrow V_{2m}$$

as the inclusion map and B as the Jacobian map. Then

$$(G(BX, BY))\, ob = g(X, Y), \quad \text{...(1.5)}$$

where g is the induced metric of V_{2m-2}. If D is the induced Riemannian connexion in V_{2m-2}, then we have the following equations

$$E_{BX}BY = B\, D_X Y + H(X, Y)M + K(X, Y)N. \quad \text{...(1.6)}$$

where M and N are unit normal vectors to V_{2m-2} and H and K are symmetric bilinear functions in V_{2m-2}.

Weingarten equations in V_{2m-2} are given by (Mishra 1972)

$$E_{BX}M = -B\,'H(X) + {}'L(X)N \tag{1.7a}$$

$$E_{BX}N = -B\,'K(X) - {}'L(X)M \tag{1.7b}$$

where

$$g('H(X), Y) \overset{def}{=} H(X, Y) \tag{1.8a}$$

$$g('K(X), Y) \overset{def}{=} K(X, Y) \tag{1.8b}$$

$'L\,(X)$ is the third fundamental tensor.

Let $'{}^*K$ and $^*K$ be the curvature tensors in V_{2m} and V_{2m-2} and 'Ric and Ric be corresponding Ricci tensors. Then from the Gauss and Mainardi-Codazzi equations (Mishra 1972)

$$('{}^*K(BX, BY, BZ, BU))ob = {}^*K(X, Y, Z, U) - H(X, U)\,H(Y, Z) + H(Y, U)H(X, Z) - K(X, U)\,K(Y, Z) + K(Y, U)\,K(X, Z) \quad ...(1.9)$$

we have

$$'\text{Ric}\,(BY, BZ) = \text{Ric}\,(Y, Z) - (C_1^1, H)\,H(Y, Z) + H('H(Y), Z), - (C_1^1\,'K)K(Y, Z) + K('K(Y), Z) \quad ...(1.10)$$

$$'r\,(BY) = Br(Y) - (C_1^1\,'H)B\,'H(Z) + {}'H('H(Y)) - (C_1^1\,'K)B\,'K(Y) + {}'K('K(Y)) \quad ...(1.11)$$

where C_1^1 is the contraction operator and

$$'\text{Ric}\,(BZ, BZ) \overset{def}{=} G\,('r(BY), BZ), \quad ...(1.12a)$$

$$\text{Ric}\,(Y, Z) \overset{def}{=} g(r(Y), Z). \quad ...(1.12b)$$

The necessary and sufficient condition that V_{2m-2} be an almost Hermite submanifold with the structure (f, g) in the almost Hermite manifold V_{2m} is (Mishra 1972)

$$F(BX) = B\overline{X} \quad ...(1.13a)$$

where

$$\overline{X} \overset{def}{=} f\,(X). \quad ...(1.13b)$$

Also when V_{2m-2} is an almost Hermite submanifold in the almost Hermite manifold V_{2m}, we have (Mishra 1972)

$$F(M) = N, \quad ...(1.14a)$$
$$F(N) = M. \quad ...(1.14b)$$

An almost Hermite submanifold V_{2m-2} in a Kähler manifold V_{2m} is Kähler and H and K of Kahler submanifold V_{2m-2} immersed in Kähler manifold V_{2m} are connected by (Mishra 1972)

$$H(X, \overline{Y}) = -K(X, Y) \text{ or } K(X, \overline{Y}) = H(X, Y). \quad ...(1.15a)$$
$$H(X, \overline{Y}) = H(\overline{X}, Y) \quad ..(1.15b)$$
$$K(X, \overline{Y}) = K(\overline{X}, Y). \quad ...(1.51c)$$

2. Submanifold

Theorem 2.1—When H and K of Kähler submanifold immersed in Kähler manifold V_{2m} are connected by (1.15a) (1.15b) and (1.15c) we also have

$$'\overline{H(X)} = 'K(X), \text{ or } 'H(X) = -'\overline{K(X)} \quad (2.1a)$$
$$'H(\overline{X}) = -'\overline{H(X)} \quad ...(2.1b)$$
$$'K(\overline{X}) = -'\overline{K(X)} \quad ...(2.1c)$$
$$'H(\overline{X}) = -'K(X) \quad ...(2.1d)$$
$$'K(\overline{X}) = 'H(X) \quad ...(2.1e)$$
$$H('H(Y), Z) = K('K(Y), Z) \quad ...(2.1f)$$
$$H('H(\overline{Y}), Z) = K('H(Y), \overline{Z} = -H('K(Y), Z) \quad ...(2.1g)$$
$$(C_1^1 {'H}) = (C_1^1 {'K}) = 0, \quad ...(2.1h)$$
$$C_1^1 ('H('K)) = C_1^1('K('H)) = 0. \quad ...(2.1i)$$

Proof : Let us take (1.15a)

$$H(X, \overline{Y}) = -K(X, Y).$$

Then

$$g('H(X), \overline{Y}) = -g('K(X), Y)$$

that is

$$g('\overline{H(X)}, Y) = g('K(X), Y).$$

Hence, we have (2.1a).

Barring (2.1a), we have $-'\overline{K(X)} = 'H(X)$.

Let us consider (1.15b)

$$H(X,\overline{Y}) = H(\overline{X}, Y).$$

Then

$$g('H(X),\ \overline{Y}) = g('H\ (\overline{X}),\ Y).$$

i.e.

$$g('\overline{H(X)},\ Y) = -\ g('H(\overline{X}),\ Y).$$

Hence, we get (2.1*b*). Similarly we can get (2.1*c*).

From (2.1*a*) and (2.1*b*), we get (2.1*d*). And from (2.1*a*) and (2.1*c*), we get (2.1*e*).

Let us take

$$K('K(X),\ Z)$$

and (2.1*a*), we have

$$K('K(X),\ Z) = K('\overline{H(X)},\ \overline{Z}).$$

Now using (1.15*c*) in this equation, we have

$$K('\overline{H(X)},\ Z) = K\ ('H(X),\ \overline{Z}).$$

Using (1.15*a*) in this, we have

$$K('\overline{H(X)},\ Z) = H('H(X),\ Z).$$

which proves (2.1*f*).

Barring Y in (2.1*f*) and using (2.1*e*), we have

$$H('H(\overline{Y}),\ Z) = K\ ('H(Y),\ Z).$$

Barring Y in (2.1*f*) and using (2.1*c*) and (1.15*c*), we have

$$\begin{aligned} H('H(\overline{Y}),\ Z) &= K('K(\overline{Y}),\ Z) \\ &= -K('\overline{K(Y)},\ Z) \\ &= -K('K(Y),\ \overline{Z}) \\ &= -H('K(Y),\ Z). \end{aligned}$$

From the above two equations, we have (2.1*g*). Contracting (2.1*b*) and (2.1*c*), we have (2.1*h*).

Theorem 2.2—Let V_{2m-2} be a Kähler submanifold in a Kähler manifold V_{2m} of constant holomorphic sectional curvature $'k$. Then in V_{2m-2}, we have

$$\begin{aligned} {}^{*}K(X,\ Y,\ Z,\ U) = {} & H(X,\ U)\ H(Y,Z) - H(X,\ Z)\ H(Y,\ U) + K(X,\ U)\ K(Y,Z) \\ & -K(X,\ Z)\ K(Y,\ U) + \tfrac{1}{4}{}'k[g(X,\ U)g(Y,\ Z) - g(X,\ Z)g(Y,\ U) + {}'f(X,\ U){}'f(Y,\ Z) \\ & -{}'f(X,\ Z)\ {}'f(Y,\ U) - 2{}'f(X,\ Y){}'f(Z,\ U)] \end{aligned} \quad \text{.. (2.2a)}$$

$$\text{Ric}\ (Y,\ Z) = \frac{'km}{2}\ g(Y,\ Z) - 2H('H(Y),\ Z) \quad \text{...(2.2b)}$$

$$R = {}'km(m-1) - 2C_1^1\ ('H('H)) \quad \text{...(2.2c)}$$

where R is the scalar curvature.

PROOF : Let us assume that the enveloping manifold V_{2m} is of constant holomorphic sectional curvature $'k$. Then the curvature tensor of V_{2m} is given by

$$\begin{aligned}('^*K(BX, BY, BZ, BU))\,ob = \tfrac{1}{4}\,'k\,[&(G(BX, BU))ob\,(G(BY, BZ))ob\\ &-(G(BX, BZ))ob\,(G(BY, BU))ob\\ &+(G(F(BX), BU))\,ob\,(G(F(BY), BZ))ob\\ &-(G(F(BX), BZ))ob\,(G(F(BY), BU))ob\\ &-2(G(F(BX), BY))ob\,(G(F(BZ), BU))ob].\end{aligned}$$

From (1.5), (1.9) and (1.13), we have (2.2*a*).

On contracting (2.2*a*) and using (2.1*f*) and (2.1*h*), we get (2.2*b*). (2.2*c*) follows from (2.2*b*).

Theorem 2.3—Let V_{2m-2} be a Kähler submanifold in a Kähler manifold V_{2m} of constant holomorphic sectional curvature $'k$. Then for H-projectively flat manifold V_{2m-2}, we have

$$H('H(X), Y) = \frac{C_1^1('H('H))}{2(m-1)}\,g(X, Y). \qquad \ldots(2.3a)$$

PROOF : The H-projective curvature tensor $'P$ in V_{2m-2} is given by (Mishra 1970)

$$\begin{aligned}'P(X, Y, Z, U) = {}^*K(X, Y, Z, U) - \frac{1}{2m}\,[&g(X, U)\,\mathrm{Ric}\,(Y, Z)\\ &- g(Y, U)\,\mathrm{Ric}\,(X, Z) + {}'f(X, U)\,\mathrm{Ric}\,(\overline{Y}, Z)\\ &- {}'f(Y, U)\,\mathrm{Ric}\,(\overline{X}, Z) - 2\,'f(Z, U)\,\mathrm{Ric}\,(\overline{X}, Y)]. \qquad \ldots(2.3b)\end{aligned}$$

Hence, making use of theorem (2.2) and (2.1) in (2.3*b*), the H-projective curvature tensor $'P$ in V_{2m-2} becomes:

$$\begin{aligned}'P(X, Y, Z, U) = {} & H(X, U)\,H(Y, Z) - H(X, Z)\,H(Y, U)\\ &+ K(X, U)\,K(Y, Z) - K(X, Z)\,K(Y, U)\\ &+ \frac{1}{m}\,[g(X, U) H('H(Y), Z) - g(Y, U)\,H('H(X), Z)\\ &+ {}'f(X, U)\,K('H(Y), Z) - {}'f(Y, U)\,K('H(X), Z)\\ &- 2\,'f(Z, U)\,K('H(X), Y)]. \qquad \ldots(2.3c)\end{aligned}$$

Let $'P(X, Y, Z, U) = 0$, then from (2.3*c*), we have

$$\begin{aligned}&H(X, U)\,'H(Y) - {}'H(X)\,H(Y, U) + K(X, U)\,'K(Y) - {}'K(X)\,K(Y, U)\\ &+ \frac{1}{m}[g(X, U)\,'H('H(Y)) - g(Y, U)\,'H('H(X)) + {}'f(X, U)\,'K('H(Y))\\ &- {}'f(Y, U)\,'K('H(X)) + 2\,\bar{U}K\,('H(X), Y)] = 0. \qquad \ldots(2.3d)\end{aligned}$$

Contracting (2.3d), we get

$$H('H(Y),U)-C_1^1('H)H(Y,U)+K('K(Y),U)-C_1^1('K)K(Y,U)$$

$$+\frac{1}{m}[g('H('H(Y)),U)-g(Y,U)C_1^1('H('H))+'f('K('H(Y),U))$$

$$-'f(Y,U)C_1^1('K('H)+2K('H(\bar{U}),Y)]=0. \qquad ..(2.3e)$$

Making use of Theorem (2.1) in (2.3e), we have

$$2(m-1)H('H(Y),(U)=C_1^1('H('H))g(Y,U) \qquad ...(2.3f)$$

which proves the statement.

Theorem 2.4—Let V_{2m-2} be a Kähler submanifold in a Kähler manifold V_{2m} of constant holomorphic sectional curvature $'k$. Then for a totally geodesic manifold V_{2m-2} we have

(a) It is H-projectively flat. (b) It is an Einstein manifold.

Hence, the manifold V_{2m-2} is of constant holomorphic sectional curvature

$$'k = R/m(m-1).$$

PROOF: Let V_{2m-2} be totally geodesic (Mishra 1965)

$$H(X,Y)=0 \text{ or } 'H(X)=0. \qquad ...(2.4a)$$

Using (2.4a) in (2.3c), we have $'P(X,Y,Z,U)=0$.

Using (2.4a) and Theorem 2.1 in Theorem 2.2, we get

$$\text{Ric}(Y,Z)=\frac{'km}{2}g(Y,Z)$$

where

$$'k=\frac{R}{m(m-1)}.$$

Hence, the proof follows.

Theorem 2.5—Let V_{2m-2} be a Kähler submanifold in H-projectively flat Kähler manifold V_{2m}. Then for H-projectively flat manifold V_{2m-2}, we have

$$H('H(X),Y)=\frac{C_1^1('H('H))g(X,Y)}{2(m-1)}.$$

PROOF: The H-projective curvature tensor $'^*P$ in V_{2m} is given by (Mishra 1970)

$$'^*P(BX,BY,BZ,BU)\,ob=('^*K(BX,BY,BZ,BU))\,ob$$

$$-\frac{1}{2(m+1)}[(G(BX,BU))\,ob\,'\text{Ric}(BY,BZ)-(G(BY,BU))ob\,'\text{Ric}(BX,BZ)$$

$$+'F(BX,BU)\,'\text{Ric}(F(BY),BZ)-'F(BY,BU)\,'\text{Ric}(F(BX),BZ)$$

$$-2'F(BZ,BU)\,'\text{Ric}(F(BX),BY)]. \qquad ...(2.5a)$$

Let $'^*P(BX, BY, BZ, BU)\, ob = 0$, then the manifold V_{2m} is on Einstein manifold

$$'\text{Ric}\,(BY, BZ) = {}'R/2m.(G(BY, BZ))ob. \qquad \ldots(2.5b)$$

Using Theorem 2.1, (1.10) and (1.11) in (2.5*b*), we get

$$\text{Ric}\,(Y, Z) + 2H('H(Y), Z) = \frac{1}{2m}[R + 2C_1^1('H('H))]\; g(Y, Z). \qquad \ldots(2.5c)$$

Contracting (2.5*c*), we get

$$R + 2C_1^1('H('H)) = 0. \qquad \ldots(2.5d)$$

Hence, we have

$$\text{Ric}\,(Y, Z) + 2H('H(Y), Z) = 0 \qquad \ldots(2.5e)$$

From (2.5*e*) and making use of Theorem 2.1 in (2.10), we get

$$'\text{Ric}\,(BY, BZ) = 0 \qquad \ldots(2.5f)$$

Hence, the curuature tensor $^*K$ in V_{2m-2} is given by

$$^*K(X, Y, Z, U) = H(X, U)H(Y, Z) - H(X, Z)\, H(Y, U) + K(X, U)\, K(Y, Z) - K(X, Z)\, K(Y, U). \qquad \ldots(2.5g)$$

Now, from (2.5*e*), (2.5*g*) and (2.3*b*), the H-projective curvature tensor $'P$ in H-projectively flat Kähler manifold V_{2m} becomes (2.3*c*). By the same method as in Theorem 2.3, we get the result.

Theorem 2.6—Let V_{2m-2} be a Kähler submanifold in a H-projectively flat Kähler manifold V_{2m}. Then for a totally geodesic manifold V_{2m-2}, we have

(*a*) It is H-projectively flat.
(*b*) It is a flat manifold.

Theorem 2.7—Let V_{2m-2} be a Kähler submanifold in a H-projectively flat Kähler manifold V_{2m}. Then for H-projectively flat submanifold V_{2m-2}, it is an Einstein manifold.

PROOF: From (2.5*e*), we have

(*a*) $\text{Ric}\,(Y, Z) = -2\, H('H'Y), Z)$.

Using Theorem 2.5 in the above, we get

(*b*) $\text{Ric}\,(Y, Z) = -\dfrac{C_1^1('H('H))}{(m-1)}\, g(Y, Z)$

which proves the statement.

Theorem 2.8—*Let* V_{2m-2} be a Kähler submanifold in a Kähler manifold V_{2m} of constant holomorphic sectional curvature $'k$. Then H-projectively flat Kähler manifold V_{2m-2} is an Einstein manifold.

PROOF: Let us consider (2.2*b*),

$$\text{Ric}\,(Y, Z) = \frac{'km}{2} g(Y, Z) - 2H('H(Y), Z).$$

Using (2.3*a*) in the above, we get

$$\text{Ric}\,(Y, Z) = \frac{R}{2(m-1)}\; g(Y, Z)$$

which proves the result.

Theorem 2.9—Let V_{2m-2} be a Kähler submanifold in a Kähler manifold V_{2m} of constant holomorphic sectional curvature $'k$. Then H-projectively flat submanifold V_{2m-2} is of constant holomorphic sectional curvature $'k$ and it is totally geodesic.

PROOF: The proof follows from the above theorems.

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