ON THE SLOW BROAD SIDE MOTION OF A THIN DISC (ELLIPTICAL) IN A VISCOUS LIQUID

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The aim of the present paper is to solve the boundary value problem relating to elliptical disc. A new infinite transform involving elliptical coordinates has been derived and some of its properties given. The drag has been calculated for small values of ellipticity.

1. INTRODUCTION

Laplace equation in three-dimensional coordinates has been solved in various systems of coordinates In the case of circular cylindrical symmetry this equation can be solved with the help of Hankel transform finite or infinite as the case may be. In many of the cases the Fourier's transform is quicker and simpler to use. Hitherto, however no such attempt has been made for the solutions of boundary value problems for elliptical discs. The aim of this paper is to supply this deficiency. The dual integral equations in elliptical coordinates have been solved for the disc.

In the first section of this investigation a new infinite transform involving elliptical coordinates has been derived and some of its properties given. In section B the boundary value problem relating to elliptical disc has been discussed in detail and the drag has been calculated for small value of ellipticity. The leading term in the value of the drag is found to be $16V\mu \cdot \frac{1}{2}h\, e^{-\xi_0}E(1-e^{-2\xi_0})^{1/2}$, E being the complete elliptical integral of the second kind, h being the distance between the foci and ξ_0 is the boundary of the ellipse. Finite transform involving Mathieu functions and its application has already been discussed by Gupta (1964) and the notations used are those as given by Mclachlan (1964).

SECTION A

2. INFINITE TRANSFORM IN ELLIPTICAL CO-ORDINATES AND THEIR PROPERTIES

Theorem 1—We know that from complex Fourier's transform extended to two variables x and y, i.e. thus if

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$$\left.\begin{aligned} \bar{f}(s,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(sx+ty)}\, dxdy \\ 4\pi^2 f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(s,t) e^{-(sx+ty)i} dsdt \end{aligned}\right] \quad \ldots 2.1)$$

then

provided

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | f(x,y) |\, dx\, dy, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | \bar{f}(s,t) |\, ds\ dt \text{ extis.}$$

Now writing

$$\left.\begin{aligned} x &= h\cosh\xi\cos\eta;\ y = h\sinh\xi\sin\eta \\ s &= h'\sinh u\cos v;\ t = h'\sinh u\sin v \end{aligned}\right\} \quad \ldots(2.2)$$

eqns. (2.1) transform into

$$\bar{f}(u,v) = \frac{h^2}{2}\int_0^{\infty}\int_0^{2\pi} f(\xi,\eta)\, [\exp ihh'\sinh u\,(\cosh\xi\cos\eta\cos v$$

$$+ \sinh\xi\sin\eta\sin v)]\,(\cosh 2\xi - \cos 2\eta) d\xi\, d\eta \quad (2.3)$$

then

$$f(\xi,\eta) = \frac{h'^2}{4\pi^2}\int_0^{\infty}\int_0^{2\pi} \bar{f}(u,v)\, [\exp - ihh'(\cosh\xi\cos\eta\ \cos v$$

$$+ \sinh\xi\sin\eta\sin v)\sinh u]\cosh u\ \sinh u\ du\, dv \quad \ldots(2\ 4)$$

provided $f(\xi,\eta)$ and $\bar{f}(u,v)$ be such that

$$\int_0^{\infty}\int_0^{2\pi} | f(\xi,\eta) || \cosh 2\xi - \cos 2\eta \mid d\xi\, d\eta \text{ and} \int_0^{\infty}\int_0^{2\pi} | \bar{f}(u,v) | \cosh u\ \sinh u\ du\, dv$$

should exist. Obviously

$$| \bar{f}(u, v) | = \left| \int_0^\infty \int_0^{2\pi} f(\xi,\eta) \Big[\exp ihh' \sinh u(\cosh \xi \cos \eta \cos v + \sinh \xi \sin \eta \sin v) \Big] (\cosh 2\xi - \cos 2\eta) \; d\xi \, d\eta \right|$$

$$\leqslant \int_0^\infty \int_0^{2\pi} | f(\xi,\eta) | | (\cosh 2\xi - \cos 2\eta) | \, d\xi \, d\eta \qquad \ldots(2.5)$$

and if the integral on the r.h.s. of (2.5) exists the integral on l.h.s. of (2.5) exists that is $f(\xi, \eta)$ $(\cosh 2\xi - \cos 2\eta)$ should be bounded in the range defined. Similarly $\bar{f}(u, v)$ should be bounded.

Property 1—Puting $x = h \cosh \xi \cos \eta = r \cos \alpha$, $y = h \sinh \xi \sin \eta = r \sin \alpha$ then $\quad r = h\,(\cosh^2 \xi - \sin^2 \eta)^{1/2}$, $\tan \alpha = \tanh \xi \tan \eta$.

Then $\exp \pm ihh' \sinh u \,(\cosh \xi \cos \eta \cos v + \sinh \xi \sin \eta \sin v)$

$$= \exp \pm i\, 2k\, (\cosh^2 \xi - \sin^2 \eta)^{1/2} \cos\, (v - \alpha).$$

Making use of the identity (3) of Mclachlan (1934, p. 43) we have

$$\exp \pm iZ_1 \cos\, (v - \alpha) = J_0(Z_1) + 2 \sum_{m=1}^{\infty} (\pm i)^m \cos m(v - \alpha)\, J_m(Z_1) \qquad \ldots(2.6)$$

where $Z_1 = 2k(\cosh^2 \xi - \sin^2 \eta)^{1/2}$; $2k = hh' \sinh u$.

Substituting (2.6) in eqns.(2.3) and (2.4) we have

$$\bar{f}(u, v) = \frac{h^2}{2} \int_0^\infty \int_0^{2\pi} f(\xi, \eta) \Big[J_0(Z'_1) + 2 \sum_{m=1}^{\infty} (i)^m J_m(Z'_1) \cos m(\eta - \alpha') \Big] \times (\cosh 2\xi - \cos 2\eta)\, d\xi \; d\eta \qquad \ldots(2.7)$$

where $Z'_1 = 2\, k(\cosh^2 \xi - \sin^2 v)^{1/2}$; $\tan \alpha' = \tanh \xi \tan v$

and

$$f(\xi, \eta) = \frac{h'^2}{4\pi^2} \int_0^\infty \int_0^{2\pi} \bar{f}(u, v) \Big[J_0(Z_1) + 2 \sum_{m=1}^{\infty} (-i)^m J_m(Z_1) \cos m(v - \alpha) \Big] \times \cosh u \sinh u \; du \; dv. \qquad \ldots(2.8)$$

Property 2—Here we shall give certain functions whose transform and inverse transform have been given.

Let us take $\quad f(\xi, \eta) = \dfrac{1}{h(\cosh \xi + \cos \eta)}$

then (2.7) gives

$$R\bar{f}(u,v) = R\int_0^\infty\int_0^{2\pi} h(\cosh\xi - \cos\eta)\exp ihh'\sinh u\left\{\cosh\xi\cos\eta\cos v + \right.$$

$$\left. + \sinh\xi\sin\eta\sin v\right\} d\xi\, d\eta$$

$$= 2\pi\int_0^\infty h\cosh\xi\, J_0(hh'\sinh u\,(\sinh^2\xi + \cos^2 v)^{1/2})d\xi$$

$$= 2\pi\cos(hh'\cos v\sinh u)/\mathrm{h}'\sinh u. \qquad \ldots(A)$$

Now inversion formula gives

$$f(\xi,\eta) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\cos(hh'\cosh u\cos v)\Big[\exp - ihh'\sinh u\,(\cosh\xi\cos\eta\cos v$$

$$+ \sinh\xi\sin\eta\sin v)\Big]\mathrm{h}'\cosh u\, du\, dv$$

$$= \text{Real}\ \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{-iw[h\,(1+\cosh\xi\cos\eta)\cos v + h\sinh\xi\sin\eta\sin v]}\ dw\, dv,$$

(where $h'\sinh u = w$)

$$= \text{Real}\ \frac{1}{2\pi}\int_0^\infty\int_{\alpha_1}^{2\pi+\alpha_1}\Big[\exp - ihw(\cosh\xi + \cos\eta)\sin\phi\Big]dw\, d\phi$$

$$= \frac{1}{2\pi}\int_0^\infty J_0(wh)\,(\cosh\xi + \cos\eta)\, dw$$

$$\left(\text{where } v + \alpha_1 = \phi \text{ and } \tan\alpha_1 = \frac{\sinh\xi\sin\eta}{1+\cosh\xi\cos\eta}\right)$$

$$= 1/h\,(\cosh\xi + \cos\eta). \qquad \ldots(B)$$

It is quite obvious that as $h \to 0$, $\xi \to \infty$, $h' \to 0$; $u \to \infty$ such that $h\cosh\xi \to r$; $h'\sinh u \to p$, (A) and (B), which are quite new results, tend to $2\pi/p$ and $1/r$ respectively, hence the transforms of $1/r$.

Transition to circular cylinder—Now if $\xi \to \infty$; $u \to \infty$, $h \to 0$, $h' \to 0$ respectively the elliptical coordinates degenerate into circular coordinates. Then $h^2\cosh 2\xi\, d\xi \to 2r\, dr$,

$h' \cosh u \to h' \sinh u \to p\ J_m(Z_1) \to J_m(Z'_1) \to J_m(pr)$ and if $f(\xi, \eta) \to \phi(r) \exp(im\,\theta)$, say, then (2.7) and (2.8) degenerate into

$$\left.\begin{aligned}\bar{\phi}(p) &= \int_0^\infty r\phi(r)\, J_m(pr)dr \\ \phi(r) &= \int_0^\infty p\bar{\phi}(p) J_m(pr)\, dp\end{aligned}\right\} \quad \ldots(2.9)$$

which is the well-known mth order Hankel transform.

Property 3—In solving the boundary value problems some simple properties of the transform are required. Let $\chi(\xi, \eta)$ be a continuous and twice differentiable function w.r.t. ξ and η with period 2π and also such that $f(\xi, \eta)$, $f'(\xi, \eta)$; $f''(\xi, \eta)$ be continuous in (ξ, η), such that

$$\left.\begin{aligned}&(i)\ \left[\chi\frac{\partial f}{\partial \xi} - f\frac{\partial \chi}{\partial \eta}\right]_0^\infty = 0 \text{ and } \left[\chi\frac{\partial f}{\partial \eta} - f\frac{\partial \chi}{\partial \eta}\right]_0^{2\pi} = 0 \\ &(ii)\ \frac{\partial^2 \chi}{\partial \xi^2} + \frac{\partial^2 \chi}{\partial \eta^2} + 2k^2(\cosh 2\xi - \cos 2\eta)\chi = 0\end{aligned}\right\} \quad \ldots(2.10)$$

where

$$\left.\begin{aligned}&\chi(\xi, \eta) = \exp ihh' \sinh u\{\cosh \xi \cos \eta \cos v + \sinh \xi \sin v \sin \eta\} \\ \text{and} \quad &4k^2 = h^2h'^2 \sinh^2 u.\end{aligned}\right\} \quad \ldots(2.11)$$

$$\text{Let} \quad \int_0^\infty \int_0^{2\pi} \left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right) \chi(\xi, \eta)\, d\xi\, d\eta = I_1 + I_2 \quad (2.12)$$

where

$$I_1 = \int_0^\infty \int_0^{2\pi} \frac{\partial^2 f}{\partial \xi^2}\chi(\xi, \eta)\, d\xi\, d\eta; \quad I_2 = \int_0^\infty \int_0^{2\pi} \frac{\partial^2 f}{\partial \eta^2}\chi(\xi, \eta) d\xi d\eta \quad \ldots(2.13)$$

Integrating I_1, by parts we have

$$I_1 = \int_0^{2\pi} \left[\chi\frac{\partial f}{\partial \xi} - f\frac{\partial \chi}{\partial \xi}\right]_0^\infty d\eta + \int_0^\infty \int_0^{2\pi} f\frac{\partial^2 \chi}{\partial \xi^2}\, d\xi d\eta$$

$$= \int_0^\infty \int_0^{2\pi} f\frac{\partial^2 \chi}{\partial \xi^2} d\xi d\eta \quad \text{[by property (i), eqn. (2.10)]}$$

and similarly

$$I_2 = \int_0^\infty \left[\chi \frac{\partial f}{\partial \xi} - f\frac{\partial \chi}{\partial \eta}\right]_0^{2\pi} d\xi + \int_0^\infty \int_0^{2\pi} f\frac{\partial^2 \chi}{\partial \eta^2}\, d\xi d\eta \,.$$

$$\text{Hence } I_1+I_2 = \int_0^\infty \int_0^{2\pi} f\left(\frac{\partial^2\chi}{\partial \xi^2}+\frac{\partial^2\chi}{\partial \eta^2}\right) d\xi d\eta$$

$$= -\frac{h^2 h'^2}{2}\int_0^\infty \int_0^{2\pi} f(\xi,\eta)\,\chi(\xi,\eta)\sinh^2 u \times (\cosh 2\xi - \cos 2\eta)\, d\xi d\eta$$

$$= -\,h'^2 \sinh^2 u\overline{f}(u, v). \qquad \ldots(2.14)$$

SECTION B

3. Application of Infinite Transform to the Motion of the Elliptical Disc

The study of slow motion of a solid body through a viscous fluid is of importance in many fields. For instance chemical engineering and viscometry are two subjects in which a quantitative understanding of such motions is essential. Hence, in recent years a systematic attempt has been made to calculate the drag force.

Though exact solutions of the linearized equations of motion are known for a considerable variety of solids moving slowly along the axis of z through an unbounded liquid. By the method of transform calculus the slow motion of a thin disc (circular) along the axis of z has been studied by Gupta (1957). But no such attempt has been made to deal with the problem of elliptical disc. The object of this investigation is to consider the slow motion of the elliptical disc moving along the axis of z.

In this section this problem has been discussed in terms of ϕ which happens to be the electrostatic potential of the plate which has been kept at a constant potential in vacuum. The function satisfies the Laplace's equation and the conditions on the boundary can be represented in terms of ϕ. The potential ϕ for an elliptical disc has been investigated by an application of the transform discussed in section A. The results obtained by Ray (1936) and Gupta (1957) are the particular cases of the results we have obtained here.

Here in this investigation the same type of integral equations have been obtained The problem is of the mixed boundary value type which has been formulated as a pair of dual integral equations. In recent years many investigators have treated such equations and there is now a number of standard methods for reducing dual relations to the solutions of a Fredholm integral equations of the second type.

In section 4 the basic formulation is given for the rigid elliptical disc translating along the axis of z. In section 5 the problem for elliptical disc has been solved by the method of the solution of dual integral equations, because of the analytic and algebraic complication involved, it is given in some detail. In section 6 the general value of drag D on the disc has been investigated and for small ellipticity the leading term for the drag has been given.

4. Basic Equations Governing the Motion

Let the elliptical disc located symmetrically be moving steadily in an unbounded viscous liquid along the axis of z with a velocity V. For sufficiently small Reynolds numbers the fluid velocity field $\mathbf{v}$ will satisfy the linearised Navier-Stokes' equations

$$\mu \text{ curl curl } \mathbf{v} = -\nabla p \qquad \ldots(4.1)$$

and

$$\text{div } \mathbf{v} = 0 \qquad \ldots(4.2)$$

μ being the coefficient of viscosity and p the dynamic pressure. Let $\mathbf{v}$ have non-vanishing components $u_1(x, y, z)$; $v_1(x, y, z)$; $w_1(x, y, z)$. We have to solve equations (4.1) and (4.2) subject to the boundary conditions,

$$\left.\begin{array}{ll} u_1 = 0 = v_1 & |z| \to \infty \\ u_1 = 0 = v_1;\ w_1 = V \text{ on } z=0,\ x^2/a^2+y^2/b^2-1<0 & \end{array}\right\} . \qquad \ldots(4.3)$$

Gupta (1957) has noted that the velocity and pressure fields $\mathbf{v}$ and p can be represented by

$$\mathbf{v} = z\nabla\phi - \phi(x, y, z)\, \mathbf{z} \qquad \ldots(4.4)$$

$$p = 2\mu \frac{\partial \phi}{\partial z} \qquad \ldots(4.5)$$

where ϕ is a harmonic function.

Thus the problem may be restated as follows:

$$\nabla^2\phi = 0 \qquad (4.6)$$

with the boundary conditions,

$$\left.\begin{array}{l} \phi = -V, \text{ within } z = 0;\ x^2/a^2 + y^2/b^2 - 1 \leqslant 0 \\ \dfrac{\partial \phi}{\partial z} = 0, \text{ outside } z = 0;\ x^2/a^2 + y^2/b^2 - 1 \geqslant 0 \end{array}\right\} \qquad \ldots(4.7)$$

5. The Case of an Elliptical Disc

Transforming equation (4.6) in elliptical coordinates, we get

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{h^2}{2}\ (\cosh 2\xi-\cos 2\eta)\ \frac{\partial^2\phi}{\partial z^2}=0 \qquad \ldots(5.1)$$

where $x = h\cosh\xi\cos\eta$, $y = h\sinh\xi\sin\eta$; $z = z$. The eqn. (5.1) is to be solved subject to the boundary condition

$$(i)\ \phi = -V;\ z = 0,\ 0\leqslant\eta\leqslant 2\pi;\ 0\leqslant\xi\leqslant\xi_0 \qquad \ldots(5.2)$$

$$(ii)\ \frac{\partial\phi}{\partial z} = 0;\ z = 0,\ 0\leqslant\eta\leqslant 2\pi,\ 0\leqslant\xi_0\leqslant\xi. \qquad \ldots(5.3)$$

Multiplying eqn. (5.1) by $\chi(\xi, \eta)$ and integrating w.r.t. ξ between 0 and ∞, and w.r.t. η between 0 and 2π where $\chi(\xi, \eta)=\exp\ ihh'\sinh u\ (\cosh\xi\cos\eta\cos v+\sinh\xi\sin\eta\sin v)$ we have

$$\int_0^\infty\int_0^{2\pi}\left[\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{h^2}{2}(\cosh 2\xi-\cos 2\eta)\frac{\partial^2\phi}{\partial z^2}\right]\chi(\xi,\eta)\,d\xi\,d\eta$$

$$=\frac{d^2\bar{\phi}}{dz^2}-h'^2\sinh^2 u\,\bar{\phi}\,(u, v, z) = 0 \qquad \ldots(5.4)$$

where

$$\bar{\phi}\,(u, v) = \frac{h^2}{2}\int_0^\infty\int_0^{2\pi}(\cosh 2\xi-\cos 2\eta)\,\phi(\xi, \eta, z)\,\chi(\xi, \eta)\,d\xi\,d\eta. \qquad \ldots(5.5)$$

The appropriate solution of (5.4) is

$$\bar{\phi}\,(u, v) = A(u, v)\exp\left(-h'\sinh u\,|\,z\,|\right)\ \text{for}\ z>0 \qquad \ldots(5.6)$$

where $A\ (u, v)$ is the unknown function to be found with the help of the boundary conditions. As the boundary conditions are of the mixed type hence we invert $\bar{\phi}$ before the boundary conditions are inverted.

In view of the axial symmetry, the solution for $\bar{\phi}$ is given by (5.6). Now inversion (as shown in section A) gives

$$\phi = \frac{h'^2}{4\pi^2}\int_0^\infty\int_0^{2\pi}A\,(u, v)\sinh u\cosh u\exp\left\{-h'.z\sinh u\right\}\left[\chi(\xi, \eta)\right]^{-1}du\,dv \qquad \ldots(5.7)$$

If ϕ is inserted in equations (5.2) and (5.3) we get the following dual integral equations:

$$\frac{h'^2}{4\pi^2}\int_0^\infty\int_0^{2\pi} A(u, v) \sinh u \cosh u \exp-\Big[ihh' \sinh u(\cosh \xi \cos \eta \cos v+ \sinh \xi \sin \eta \sin v)\Big] du\, dv = -V \qquad \text{...(5.8)}$$

$$(\text{for } 0\leqslant\xi\leqslant\xi_0;\ z = 0,\ 0\leqslant\eta\leqslant 2\pi)$$

and

$$\frac{h'^2}{4\pi^2}\int_0^\infty\int_0^{2\pi} A(u, v)\, h'\sinh^2 u \cosh u \exp-\Big[ihh' \sinh u\, (\cosh \xi \cos \eta \cos v + \sinh \xi \sin \eta \sin v)\Big] du\, dv = 0$$

$$(\text{for } \xi>\xi_0,\ 0\leqslant\eta\leqslant 2\pi,\ z = 0). \qquad (5.9)$$

Before solving the above equations we expand

$\exp -ihh' \sinh u\ (\cosh \xi \cos \eta \cos v+ \sinh \xi \sin \eta \sin v)$.

From eqn. (2.6) we have

$$\Big[\chi(\xi, \eta)\Big]^{-1} = \exp -i\Big[\frac{hh' \sinh u}{2}\Big\{e^{2\xi}+e^{-2\xi}+2 \cos 2\eta\Big\}^{1/2} \cos (v-\alpha)\Big]$$

$$= J_0\Big\{\frac{hh' \sinh u}{2}(e^{2\xi}+e^{-2\xi}+2 \cos 2\eta)^{1/2}\Big\}$$

$$+2\sum_{m=1}^{\infty}(-i)^m J_m\Big\{\frac{hh' \sinh u}{2}(e^{2\xi}+e^{-2\xi}+2\cos 2\eta)^{1/2}\Big\} \cos m(v-\alpha). \qquad \text{...(5.10)}$$

Following Watson (1952, pp. 359, 360), we have

$$J_0\Big[\frac{hh' \sinh u}{2}(e^{-2\xi}+e^{2\xi}+2 \cos 2\eta)^{1/2}\Big]$$

$$= \sum_{q=0}^{\infty}(-1)^q \varepsilon_q J_q\, (\tfrac{1}{2}hh' \sinh u.e^{\xi})\, J_q(\tfrac{1}{2}hh' e^{-\xi} \sinh u) \cos 2q\eta \qquad \text{...(5.11)}$$

where $\varepsilon_0 = 1$, $\varepsilon_q = 2$ for $q\geqslant 1$.

Substituting the value of $[\chi(\xi,\eta)]^{-1}$ from (5.11) in (5.8) and (5.9) we have,

$$\frac{h'^2}{4\pi^2}\int_0^\infty\int_0^{2\pi} A(u,v)\sinh u\cosh u\Bigg[\Bigg\{\sum_{q=0}^{\infty}(-1)^q\,\varepsilon_q J_q\left(\frac{h'\sinh u.h\,e^{\xi}}{2}\right)$$

$$\times J_q\left(\frac{hh'\sinh u.e^{-\xi}}{2}\right)\cos 2q\eta\Bigg\}+2\sum_{m=1}^{\infty}(-i)^m J_m\Bigg\{\frac{hh'\sinh u}{2}(e^{2\xi}+e^{-2\xi}$$

$$+2\cos 2\eta)^{1/2}\cos m(v-\alpha)\Bigg\}\Bigg]du\,dv=-V$$

$$(\text{for } z=0,\ \ 0\leqslant\eta\leqslant 2\pi,\ 0<\xi<\xi_0) \qquad ...(5.12)$$

and

$$\frac{h'^3}{4\pi^2}\int_0^\infty\int_0^{2\pi} A(u,v)\sinh^2 u\cosh u\Bigg[\Bigg\{\sum_{q=0}^{\infty}(-1)^q\,\varepsilon_q J_q\left(\frac{h'\sinh u.\,he^{\xi}}{2}\right)$$

$$J_q\left(\frac{h'\sinh u.\,he^{-\xi}}{2}\right)\cos 2q\eta\Bigg]+2\Bigg\{\sum_{m=1}^{\infty}(-i)^m$$

$$\times J_m\left[\frac{hh'\sinh u}{2}(e^{2\xi}+e^{-2\xi}+2\cos 2\eta)^{1/2}\right]\cos m(v-\alpha)\Bigg\}\Bigg]du\,dv=0$$

$$(\text{for } z=0;\ \xi>\xi_0;\ 0\leqslant\eta\leqslant 2\pi). \qquad ...(5.13)$$

Following Sneddon (1966, pp. 129, 130) and Watson (1952, p. 411) we next represent

$$A(t,v)=-Bt^{-3/2}J_{1/2}(at) \qquad ...(5.14)$$

where $h\cosh\xi_0=a$, $h'\sinh u=t$ which ensures that eqns. (5.13) is automatically satisfied (Watson 1952, p. 411).

Since for $\xi>\xi_0$, $a<(b+c)$ i.e. $h\cosh\xi_0<h\left(\frac{e^{\xi}+e^{-\xi}}{2}\right)$ if $\xi>\xi_0$.

If we substitute from (5.14) into equation (5.12) and integrate both the sides with respect to η between 0 and 2π we have

$$\frac{B}{2\pi}\int_0^\infty J_{1/2}(at)J_0\left(\frac{he^{\xi}t}{2}\right)J_0\left(\frac{h}{2}e^{-\xi}t\right)t^{1/2}\,dt=-V \qquad ...(5.15)$$

$$(\text{for } \xi_0>\xi,\ z=0)$$

which following Watson [1952, p. 419, eqn. (6)] gives

$$B = 2\sqrt{2\pi a}V. \qquad \ldots(5.16)$$

Thus the required solution is

$$\phi = -\int_0^\infty 2\sqrt{2\pi a}\, V J_{1/2}(at)\, t^{-1/2} \sum_{q=0}^{\infty} (-1)^q \varepsilon_q J_q\left(\frac{he^{\xi}t}{2}\right) J_q\left(\frac{he^{-\xi}t}{2}\right)$$

$$\times \cos 2q\eta\, e^{-t|z|}\, dt$$

or

$$\phi = -2\sqrt{2\pi a}\, V \int_0^\infty J_{1/2}(ah'\sinh u).\, h' \cosh u\, (h' \sinh u)^{-1/2}\, e^{-h' \sinh u|z|}$$

$$\times J_0\left[\frac{hh' \sinh u}{\sqrt{2}} (\cosh 2\xi + \cos 2\eta)^{1/2}\right] du \qquad \ldots(5.17)$$

which is quite a new result for the elliptical disc.

When $h \to 0$; $\xi \to \infty$ (we have for a circular cylinder)

$$\phi = -\frac{2V}{\pi} \int_0^\infty e^{-pz} J_0(rp) \frac{\sin p}{p}\, dp.$$

6. Drag

Here in this section we have calculated the drag on the elliptical disc. For the drag calculation the relevant stress component is $\widehat{zz}$ where

$$\widehat{zz} = -p + 2\mu \left(\frac{\partial \phi}{\partial z}\right) \qquad \ldots(6.1)$$

which in virtue of (4.5) on $z = 0$ gives

$$\widehat{zz} = -2\mu \left(\frac{\partial \phi}{\partial z}\right)_{z=0\pm}$$

and the drag force on the elliptical disc is

$$D = 2\mu \int_S \left\{\left[\widehat{zz}\right]_{0+} - \left[\widehat{zz}\right]_{0-}\right\} dS$$

$$D=\frac{2\mu h^2B}{2\pi}\int_0^{\xi_0}\int_0^{2\pi}(\cosh 2\xi-\cos 2\eta)\left[\int_0^{\infty}t^{-1/2}J_{1/2}\,(h\cosh\xi_0 t)\right.$$

$$\left.\times J_0\,(th\,(\cosh^2\xi-\sin^2\eta)^{1/2}\,dt\right]d\xi\,d\eta$$

$$=\frac{\mu h^2B}{\pi}\sqrt{\left(\frac{2}{\pi a}\right)}\int_0^{\xi_0}\int_0^{2\pi}\frac{\cosh 2\xi-\cos 2\eta}{(h^2\cosh^2\xi_0-h^2\cosh^2\xi+h^2\sin^2\eta)^{1/2}}\,d\xi\,d\eta$$

$$=\frac{4h^2V\mu}{\pi}\int_0^{\xi_0}\int_0^{2\pi}\frac{\cosh 2\xi-\cos 2\eta}{\sqrt{h^2(\cosh^2\xi_0-\cosh^2\xi)}}\left[1+\frac{\sin^2\eta}{(\cosh^2\xi_0-\cosh^2\xi}\right]^{-1/2}d\xi\,d\eta$$

$$=\frac{4\sqrt{2}hV\mu}{\pi}\int_0^{\xi_0}\int_0^{2\pi}\frac{\cosh 2\xi-\cos 2\eta}{h\sqrt{(\cosh 2\xi_0-\cosh 2\xi)}}\sum_{s=0}^{\infty}\left(\frac{\sin^2\eta}{\cosh^2\xi_0-\cosh^2\xi}\right)^s\frac{(-1/2)s}{s!}\,d\xi\,d\eta$$

$$=8\sqrt{2}hV\mu\left[\int_0^{\xi_0}\frac{\cosh 2\xi\,d\xi}{(\cosh 2\xi_0-\cosh 2\xi)^{1/2}}-\frac{1}{2}\int_0^{\xi_0}\frac{(\cosh 2\xi+\frac{1}{2})\,d\xi}{(\cosh 2\xi_0-\cosh 2\xi)^{3/2}}+\ldots\right].\qquad\ldots(6.2)$$

For small values of h the leading term in (6.2) can be evaluated as follows:

$$D=4\mu\sqrt{2}hV\int_0^{v_0}\frac{\cosh v}{(\cosh v_0-\cosh v)^{1/2}}\,dv,\ \text{where } 2\xi=v$$

$$=4\pi\mu hV\,P_{1/2}\,(\cosh 2\xi_0)$$

$$=8\pi\mu hV.\frac{1}{\pi}\,e^{\xi_0}\;E\left[(1-e^{-2\xi_0})^{1/2}\right]$$

$$=8\mu hV\,e^{\xi_0}\;E((1-e^{-2\xi_0})^{1/2})$$

where E is the complete integral of the second kind.
When $h\to 0$ and $\xi_0\to\infty$ the elliptic disc degenerates into a circular disc and the drag is

$$D=16\mu Va$$

which is in complete agreement with that of Gupta (1957).

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