ON SASAKIAN MANIFOLD WITH RECURRENT CURVATURE TENSORS

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In this paper we have obtained some birecurrent properties of C-Bochner curvature tensor, projective curvature tensor, conformal curvature tensor and conharmonic curvature tensor in a Sasakian manifold. Further it is proved that the conharmonic and conformal symmetric Sasakian manifold is an C-Einstein manifold.

1. INTRODUCTION

Let us consider an n $(= 2m + 1)$ dimensional real differentiable manifold V_n. Let there exist a vector valued linear function F, a C^∞ vector field T and a C^∞ 1-form A satisfying

$$\bar{\bar{X}} + X = A(X)T, \text{ for arbitrary vector field } X \qquad \ldots(1.1a)$$

where

$$F(X) \overset{def}{=} \bar{X} \qquad \ldots(1.1b)$$

$$A(T) = 1 \qquad \ldots(1.2)$$

$$\bar{T} = 0 \qquad \ldots(1.3)$$

$$A(\bar{X}) = 0 \qquad \ldots(1.4)$$

the rank of $n \times n$ matrix $((F))$ is $(n-1)$. ...(1.5)

Then the manifold V_n is said to have an almost contact structure (F, T, A) and V_n is called an almost contact manifold.

Agreement 1.1 — The equations containing X, Y, Z, U, V will hold for arbitrary vector fields X, Y, Z, U, V in V_n.

Let the almost contact manifold V_n be endowed with the non-singular metric tensor g. Let us put

$$g(\bar{X}, Y) = {'F}(X, Y). \qquad \ldots(1.6)$$

If the following equations hold

$$'F(X, Y) + {'F}(Y, X) = 0 \qquad \ldots(1.7)$$

$$g(T, X) = A(X) \quad ...(1.8)$$

$$g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)A(Y) \quad ...(1.9)$$

then the manifold V_n is called an almost contact metric manifold or an almost Grayan manifold, (F, T, A, g) being an almost Grayan structure.

Let in an almost Grayan manifold

$$2\ 'F(X, Y) = (D_X A)(Y) - (D_Y A)(X) \quad ...(1.10)$$

$$(D_X\ 'F)(Y, Z) + (D_Y\ 'F)(Z, X) + (D_Z\ 'F)(X, Y) = 0 \quad ...(1.11)$$

hold where D is the Riemannian connexion. Then the manifold V_n is called an almost Sasakian manifold, (F, T, A, g, D) being an almost Sasakian structure.

If in an almost Sasakian manifold A is Killing

$$(D_X A)(Y) + (D_Y A)(X) = 0 \quad ...(1.12)$$

the manifold is called K-contact Riemannian manifold. If in the K-contact Riemannian manifold

$$(D_X F)(Y) = A(Y)X - g(X, Y)T$$

is satisfied then the manifold is called a Sasakian manifold. Thus in a Sasakian manifold, we have

$$'F(X, Y) = (D_X A)(Y) \quad ...(1.13)$$

$$(D_Z\ 'F)(X, Y) = 'K(X, Y, Z, T) \quad ...(1.14a)$$

where

$$'K(X, Y, Z, T) \overset{def}{=} g(K(X, Y, Z), T) \quad ...(1.14b)$$

and K is the curvature tensor of D.

$$'K(X, Y, Z, T) = A(X)\, g(Y, Z) - A(Y)\, g(X, Z) \quad ...(1.15)$$

$$K(T, Y, Z) = g(Y, Z)T - A(Z)Y \quad ...(1.16)$$

$$K(X, Y, T) = A(Y)X - A(X)Y \quad ...(1.17)$$

$$\text{Ric}\,(\bar{X}, Y) + \text{Ric}\,(X, \bar{Y}) = 0 \quad ...(1.18)$$

$$\text{Ric}\,(X, T) = 2m\, A(X) \quad ...(1.19a)$$

where Ric is the Ricci tensor defined by

$$\text{Ric}\,(Y, Z) = (C_1^1\, K)(Y, Z) \qquad \text{...(1.19}b\text{)}$$

and C_1^1 is contraction in the first slot.

The C-Bochner curvature tensor B in a Sasakian manifold V_n is given by (Matsumoto and Chuman 1969)

$$\begin{aligned} B(X, Y, Z) \overset{def}{=} K(X, Y, Z) &+ \frac{1}{(n+3)}\Big[\text{Ric}\,(X, Z)Y - \text{Ric}\,(Y, Z)X + g(X, Z)\, r(Y) \\ &- g(Y, Z)\, r(X) + \text{Ric}\,(\bar{X}, Z)\bar{Y} - \text{Ric}\,(\bar{Y}, Z)\bar{X} + {'F}(X, Z)\, r(\bar{Y}) \\ &- {'F}(Y, Z)\, r(\bar{X}) + 2\,\text{Ric}\,(\bar{X}, Y)\bar{Z} + 2\,{'F}(X, Y)\, r(\bar{Z}) - \text{Ric}\,(X, Z)A(Y)T \\ &+ \text{Ric}\,(Y, Z)\, A(X)T - A(X)\, A(Z)\, r(Y) + A(Y)\, A(Z)\, r(X)\Big] \\ &- \frac{(k+n-1)}{(n+3)}\Big[{'F}(X, Z)\bar{Y} - {'F}(Y, Z)\bar{X} + 2\,{'F}(X, Y)\bar{Z}\Big] \\ &+ \frac{k}{(n+3)}\Big[g(X, Z)A(Y)T + A(X)A(Z)Y - g(Y, Z)A(X)T - A(Y)A(Z)X\Big] \\ &- \frac{(k-4)}{(n+3)}\Big[g(X, Z)Y - g(Y, Z)X\Big] \end{aligned} \qquad \text{...(1.20)}$$

where

$$k = \frac{R+n-1}{n+1}, \quad \text{Ric}\,(Y, Z) \overset{def}{=} g(r(Y), Z) \qquad \text{...(1.21)}$$

and R is the scalar curvature $(C_1^1\, r)$.

Also the projective tensor W, the conformal tensor V^* and conharmonic tensor L are given by (Mishra 1970):

$$W(X, Y, Z) \overset{def}{=} K(X, Y, Z) - \frac{1}{2\,m}\Big[\text{Ric}\,(Y, Z)X - \text{Ric}\,(X, Z)Y\Big] \qquad \text{...(1.22)}$$

$$\begin{aligned} V^*(X, Y, Z) \overset{def}{=} K(X, Y, Z) &- \frac{1}{(2m-1)}\Big[\text{Ric}\,(Y, Z)X - \text{Ric}\,(X, Z)Y \\ &+ g(Y, Z)\, r(X) - g(X, Z)\, r(Y)\Big] + \frac{R}{2m(2m-1)}\Big[g(Y, Z)X - g(X, Z)Y\Big] \end{aligned} \qquad \text{...(1.23)}$$

$$\begin{aligned} L(X, Y, Z) \overset{def}{=} K(X, Y, Z) &- \frac{1}{(2m-1)}\Big[g(Y, Z)\, r(X) - g(X, Z)\, r(Y) \\ &+ \text{Ric}\,(Y, Z)X - \text{Ric}\,(X, Z, Y)\Big] \end{aligned} \qquad \text{...(1.24)}$$

respectively.

2. Recurrent Curvature Tensors

Definition 2.1 — The Sasakian manifold will be called recurrent in P with α as the parameter of recurrence if

$$(D_Y P)(Z, U, V) = \alpha(Y)\, P(Z, U, V) \qquad \ldots(2.1)$$

where P is any of the curvature tensors.

Definition 2.2 — The Sasakian manifold will be called birecurrent in P with β as the parameter of recurrence if

$$(D_X D_Y P)(Z, U, V) - (\nabla P)(Z, U, V, D_X Y) = \beta(Y, X)\, P(Z, U, V) \qquad \ldots(2.2)$$

equivalent to

$$(\nabla \nabla P)(Z, U, V, Y, X) = \beta(Y, X)\, P(Z, U, V)$$

where

$$D_X Y \overset{def}{=} (\nabla Y)(X).$$

The Sasakian manifold is said to be symmetric in P if

$$(D_Y P)(Z, U, V) = 0. \qquad \ldots(2.3)$$

The Sasakian manifold is said to be C-Einstein manifold (Matsumoto and Chuman 1969) if

$$\text{Ric}\,(U, V) = a\, g(U, V) + b\, A(U)\, A(V) \qquad \ldots(2.4)$$

where a and b are some scalars.

Theorem 2.1 — Every recurrent Sasakian manifold for which the recurrence parameter α satisfies

$$(D_X\, \alpha)(Y) + \alpha(X)\, \alpha(Y) \not\equiv 0 \qquad \ldots(2.5)$$

is a birecurrent Sasakian manifold but the converse is not true necessarily.

Proof : From (2.1), we have

$$(\nabla \nabla P)(Z, U, V, Y, X) = ((D_X\, \alpha)(Y) + \alpha(X)\, \alpha(Y))\, P(Z, U, V) \qquad \ldots(2.6)$$

Comparing (2.6) and (2.2), we get

$$\beta(X, Y) = (D\alpha)(Y) + \alpha(X)\, \alpha(Y)$$

which proves the statement.

Theorem 2.2 — In a Sasakian manifold, we have

$$A(X)\, A(P(Y, U, V)) + A(Y)\, A(P(Z, X, V)) + A(Z)\, A(P(X, Y, V)) = 0 \qquad \ldots(2.7)$$

$$A(P(\bar{Z}, \bar{U}, V)) = 0 \qquad \ldots(2.8)$$

$$A(P(\bar{Z}, T, V)) + A(P(T, \bar{Z}, V)) = 0 \qquad \ldots(2.9)$$

$$A(P(T, \bar{U}, V) + A(P(T, U, \bar{V})) = 0. \qquad \ldots(2.10)$$

PROOF : From (1.22), we have

$$A(W(Z, X, V) = A(K(Z, X, V)) - \frac{1}{(n-1)}\Big[A(Z) \operatorname{Ric}(X, V) - A(X) \operatorname{Ric}(Z, V)\Big] \qquad \ldots(2.11)$$

Multiplying (2.11) by $A(Y)$ and writing two other equations by taking cycle permutation of Z, U and Y and adding we get (2.7). Equation (2.8) follows from (2.11) by barring Z and U in (2.11) and using (1.4) and (1.15). The proof of (2.9) and (2.10) is obvious from (2.11) for projective curvature tensor W. The proof for other curvature tensor will follow similarly.

Theorem 2.3 — For a P-birecurrent Sasakian manifold, we have

$$'P(Z, U, V, Y) + A(V)\, A(P(Z, U, Y)) - A(P(T, U, V))\, g(Y, Z) - A(P(Z, T, V))\, g(Y, U) = \delta(Y, T)\, A(P(Z, U, V)) \qquad \ldots(2.12a)$$

where

$$\delta(Y, X) \overset{def}{=} \beta(Y, X) - \beta(X, Y) \qquad (2.12b)$$

and

$$'P(Z, U, V, Y) \overset{def}{=} g(P(Z, U, V), Y). \qquad \ldots(2.12c)$$

PROOF : Let the manifold V_n be P-birecurrent. Then from (2.2), we get

$$(\nabla\nabla P)(Z, U, V, Y, X) - (\nabla\nabla P)(Z, U, V, X, Y) = \delta(Y, X)\, P(Z, U, V) \qquad \ldots(2.13)$$

Consequently from (2.13) by virtue of Ricci identity, we have

$$K(X, Y, P(Z, U, V)) - P(K(X, Y, Z), U, V) - P(Z, K(X, Y, U), V) - P(Z, U, K(X, Y, V)) = \delta(Y, X)\, P(Z, U, V). \qquad \ldots(2.14)$$

This equation implies

$$'K(T, Y, P(Z, U, V), T) - 'P(K(T, Y, Z), U, V, T) - 'P(Z, K(T, Y, U), V, T) - 'P(Z, U, K(T, Y, V), T) = \delta(Y, T)\, 'P(Z, U, V, T). \qquad \ldots(2.15)$$

Using (1.8), (1.14), (1.15) and (1.16) in (2.15), we obtain

$$g(P(Z, U, V), Y)-A(Y)\, A(P(Z, U, V))-'P(T, U, V, T)\, g(Y, Z) + A(Z)\, 'P(Y, U, V, T)-'P(Z, T, V, T)\, g(Y, U) + A(U)\, 'P(Z, Y, V, T) - 'P(Z, U, T, T)\, g(Y, V) + A(V)\, 'P(Z, U, Y, T) = \delta(Y, T)\, 'P(Z, U, V, T). \quad ...(2.16)$$

Making use of (2.7) in (2.16), we get (2.12).

Theorem 2.4 — *K*-birecurrent Sasakian manifold is of constant Riemannian curvature.

PROOF : For birecurrent Sasakian manifold from (2.16), we get

$$K(Z, U, Y) = g(Y, U)Z-g(Y, Z)U + \delta(Y, T)(A(U)Z-A(Z)U. \quad ...(2.17)$$

Contracting (2.17), we get

$$\mathrm{Ric}\,(U, Y) = (n-1)\,(g(U, Y)+ \delta(Y, T)\, A(U) \quad ...(2.18)$$

On the other hand Takano (1971) proved the identity

$$\delta(X, Y)\,\mathrm{Ric}\,(U, Z) + \delta(Y, U)\,\mathrm{Ric}\,(X, Z) + \delta(U, X)\,\mathrm{Ric}\,(Y, Z) = 0. \quad ...(2.19)$$

Putting T for Z and T for U in (2.19), we have

$$\delta(X, Y) + \delta(Y, T)\, A(X) + \delta(T, X)\, A(Y) = 0. \quad ...(2.20)$$

Using symmetric property of Ric and g in (2.18), we get

$$\delta(Y, T)\, A(U) = \delta(U, T)\, A(Y). \quad ...(2.21)$$

Substituting from (2.21) in (2.20), we get

$$\delta(X, Y) + A(Y)(\delta(X, T) + \delta(T, X)) = 0 \quad ...(2.22)$$

which implies

$$\delta(X, Y) = 0. \quad ...(2.23)$$

Consequently from (2.17) and (2.13) the statement follows.

Corollary 2.1 — In a birecurrent Sasakian manifold the scalar valued function β is symmetric :

$$\beta(X, Y) = \beta(Y, X).$$

PROOF : It is obvious from (2.23) and (2.12*b*).

Theorem 2.5 — The projective birecurrent, conformal birecurrent, conharmonic birecurrent and *C*-Bochner birecurrent Sasakian manifolds are projectively flat, conformally flat, conharmonically flat and *C*-Bochner flat respectively.

PROOF : Barring Z and U in (2.12) and using (2.8), we get

$$'P(Z, \bar{U}, V, Y) = 'F(Z, Y)\, A(P(T, \bar{U}, V)) + 'F(U, Y)\, A(P(\bar{Z}, T, V)) \quad \ldots(2.24)$$

Barring V and Y in (2.12) and using (1.4), we get

$$'P(Z, U, \bar{V}, \bar{Y}) = 'F(Y, Z)\, A(P(T, U, \bar{V}) + A(P(Z, T, \bar{V}))\, 'F(Y, U) + \delta(\bar{Y}, T)\, A(P(Z, U, V)). \quad \ldots(2.25)$$

Subtracting (2.24) from (2.25) and using (1.7), (2.9) and (2.10), we get

$$'P(Z, U, \bar{V}, \bar{Y}) - 'P(\bar{Z}, \bar{U}, V, Y) = \delta(\bar{Y}, T)\, A(P(Z, U, \bar{V})). \quad \ldots(2.26)$$

Using Theorem 2.1 of Rathore and Mishra (1976) in (2.26), we get

$$A(P(Z, U, \bar{V})) = 0 \Rightarrow A(P(Z, U, V) = 0. \quad \ldots(2.27)$$

Since δ is non-vanishing. Substituting (2.27) in (2.12), wet get

$$'P(Z, U, V, Y) = 0,$$

which proves the statement.

Theorem 2.6 — A conformal or conharmonic symmetric Sasakian manifold is an C-Einstein manifold

$$\text{Ric}\,(U, V) = b_1 g(U, V) + b_2\, A(U)\, A(V) \quad \ldots(2.28)$$

where

$$b_1 = \frac{R-(n-1)}{(n-1)}, \quad b_2 = -\frac{R-n(n-1)}{(n-1)} \text{ and } b_1 + b_2 = n-1,$$

while the curvature tensor $'K$ is given by

$$'K(Z, U, V, Y) = (1 + b_3)\,[g(Z, Y)\, g(U, Y) - g(Z, V)\, g(U, Y)] - b_3[g(Z, Y)\, A(U)\, A(Y) + g(U, Y)\, A(Z)\, A(Y) - g(Z, V)\, A(U)\, A(Y) - g(U, Y)\, A(Z)\, A(V)] \quad \ldots(2.29a)$$

where

$$b_3 \overset{def}{=} -\frac{b_2}{n-2}. \quad \ldots(2.29b)$$

PROOF : Putting $\delta = 0$ in (2.12) for conformal symmetric Sasakian manifold, we have

$$(n-2)\, V^*\,(Z, U, V) = A(V)\,(A(Z)\, r(U) - A(U, r(Z)) + \left[\left(1 - \frac{R}{(n-1)}\right) g(Z, V) + \text{Ric}\,(Z, V) - (n-1)\, A(V)\, A(Z)\right] U - \left[\left(1 - \frac{R}{(n-1)}\right) g(U, V) + \text{Ric}\,(U, V) - (n-1)\, A(U)\, A(V)\right] Z \quad \ldots(2.30)$$

From (2.30) and (1.23), we have

$$(n-2)\, K(Z, U, V) = [g(U, V) - A(U)\, A(V)]\, r(Z)$$
$$+ [g(Z, V) - (n-1)\, A(Z)\, A(V)]\, U - [g(Z, V) - A(Z)\, A(V)]\, r(U)$$
$$- [g(U, V) - (n-1)\, A(U)\, A(V)]\, Z. \quad \ldots(2.31)$$

Contracting (2.31), we get

$$(n-1) \text{ Ric } (U, V) = (R-n+1)\, g(U, V) + (n(n-1)-R)\, A(U)\, A(V). \quad \ldots(2.32)$$

From (2.32), we have

$$(n-1)\, r(U) = (R-n+1)U + (n(n-1)-R)\, A(U)T. \quad \ldots(2.33)$$

From (2.33) and (2.31), we get (2.29).

The proof for conharmonic symmetric Sasakian manifold will follow similarly.

Corollary 2.2 — The projective birecurrent Sasakian manifold is of constant Riemannian curvature.

PROOF : It is obvious from Theorem 2.5.

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