# STRONG-CUT CUTTING PLANE PROCEDURE FOR EXTREME POINT MATHEMATICAL PROGRAMMING PROBLEMS

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In this paper a strong cut procedure is developed for solving an extreme point mathematical programming problem :

Max $Z = CX$

subject to $AX = b$

and $X$ is an extreme point of

$DX = d,\ X \geqslant 0.$

The algorithm does away with the calculation of some of the extreme points of $DX = d,\ X \geqslant 0$ and thus moves towards optimality at a much faster rate. This procedure will be very efficient computationally. A numerical example is also solved to illustrate the procedure.

## INTRODUCTION

The most general form of extreme point mathematical programming problem is :

$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } AX = b \\ \text{and } X \text{ is an extreme point of} \\ DX = d \\ X \geqslant 0 \end{array}\right\} \qquad \text{...Problem (I)}$$

where $A$ is $m \times n$, $X$ is $n \times 1$, $b$ is $m \times 1$, $C$ is $1 \times n$, $D$ is $p \times n$, $d$ is $p \times 1$ and 0 is $n \times 1$.

A cutting plane algorithm was established by Kirby *et al.* (1972a) in which one has to test the linear independence of a sub-set of columns of $D$ at each iteration and also at each stage alternative optima are needed. Kirby *et al.* (1972b) presented another method in which one need not examine the linear independence of a sub-set of columns of $D$ and in its place one has to test for feasibility of extreme points of

$DX = d,\ X \geqslant 0$ with respect to $AX = b$, but alternative optima were still needed at each step. Puri and Swarup (1973) further improved upon the methods of Kirby *et al.* (1972a, 1972b) by introducing a deep cut to avoid investigation of some of the extreme points of $DX = d,\ X \geqslant 0$. In the present study, the method is further improved upon by introducing still deeper cut and we shall call this as a 'Strong Cut'.

## Theoretical Development

To solve problem (I), we will deal with two problems :

$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } FX = f \\ \qquad\qquad X \geqslant 0 \\ \text{where } F = \begin{bmatrix} A \\ D \end{bmatrix}, f = \begin{bmatrix} b \\ d \end{bmatrix} \end{array}\right\} \quad \ldots\text{Problem (II.1)}$$

and

$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } DX = d \\ \qquad\qquad X \geqslant 0. \end{array}\right\} \quad \ldots\text{Problem (II.2)}$$

Original problem (I) is always bounded since its solution is to be an extreme point of $DX = d,\ X \geqslant 0$ and these extreme points are finite in number. Problems (II.1) and (II.2) can be bounded or unbounded. If they are unbounded, they can be made bounded by introducing a constraint $CX \leqslant M$, $M$ being sufficiently large positive number, without losing any of the extreme points of the problems to which the constraint is introduced.

## Notations

$S_1$ = set of extreme points of (II.1)
$= [X/X$ is an extreme point of $FX = f,\ X \geqslant 0]$

$S_2$ = set of extreme points of (II.2)
$= [X/X$ is an extreme point of $DX = d,\ X \geqslant 0]$

$S$ = set of feasible points for (I) $= [X \in S_2/AX = b]$

$X_{\binom{1}{1}}$ = set of optimal extreme points of (II.1)

$$= \left[X^1_{1\,1}, X^1_{1\,2}, \ldots\ldots, X^1_{1\,s_1}\right]$$

$X_{\binom{1}{2}}$ = set of second best extreme points of (II.1)

$$= \left[X^1_{2\,1}, X^1_{2\,2}, \ldots\ldots, X^1_{2\,s_1}\right]$$

(see definition of second best extreme point solution in appendix. For ready reference, method of finding it is also given in appendix)

$u_{\binom{1}{1}}$ = value of the objective function at optimal extreme points of (II.1)

$= CX^1_{11}$

$u_{\binom{1}{2}}$ = value of the objective function at elements of $X_{\binom{1}{2}}$

$= CX^1_{21}$

$X_{\binom{2}{1}}$ = set of optimal extreme points of (II.2)

$= \left[X^2_{11}, X^2_{12}, \ldots\ldots, X^2_{1k_1}\right]$

$u_{\binom{2}{1}}$ = value of the objective function at elements of $X_{\binom{2}{1}}$

$= CX^2_{11}$

$\phi$ = null vector.

## Theorem

Every extreme point of $DX = d$, $X \geqslant 0$ satisfying feasibility in $AX = b$ is also an extreme point of $FX = f$, $X \geqslant 0$. But there may be extreme points of $FX = f$, $X \geqslant 0$ which are not extreme point of $DX = d$, $X \geqslant 0$. That is $S \subseteq S_1$ (Kirby *et al.* 1972a, Puri and Swarup 1973).

## Procedure

Using simplex method find $X_{\binom{1}{1}}$ assuming, of course, that $S_1 \neq \phi$ because if $S_1 = \phi$ then problem (II.1) and hence problem (I) will have no solution (Kirby *et al.* 1972a). If $X_{\binom{1}{1}} \cap S \neq \phi$, then every $X$ belonging to $X_{\binom{1}{1}} \cap S$ will be the required optimal solution of (I) yielding optimal value as $u_{\binom{1}{1}}$ (Kirby *et al.* 1972a). $X_{\binom{1}{1}} \cap S$ will be non-null iff there exists at least one element of $X_{\binom{1}{1}}$ which is such that the number of non-null columns of $D$ corresponding to non-zero basic variables in that element is less than or equal to $p$ and these non-null columns are linearly independent. $X_{\binom{1}{1}} \cap S$ will be $\phi$ if (i) the number of non-null columns of $D$ corresponding to non-zero basic variables in any element of $X_{\binom{1}{1}}$ is greater than $p$ or (ii) the number is less than or equal to $p$ but they are not linearly independent. Thus elements of $X_{\binom{1}{1}}$ are tested one by one to see whether any one of them is an extreme point of $DX = d$, $X \geqslant 0$ or not (Kirby *et al.* 1972a).

If $X_{\binom{1}{1}} \cap S = \phi$, we proceed to find $X_{\binom{1}{2}}$ and $u_{\binom{1}{2}}$ (Hadley 1962 and Kirby *et al.* 1972a). For ready reference, method of finding second best extreme point solutions is given in appendix. If $X_{\binom{1}{2}} = \phi$, (I) has no solution. If $X_{\binom{1}{2}} \cap S \neq \phi$, then every $X \in X_{\binom{1}{2}} \cap S$ will be an optimal solution of (I). If $X_{\binom{1}{2}} \neq \phi$ but $X_{\binom{1}{2}} \cap S = \phi$ (i.e. no element of $X_{\binom{1}{2}}$ is an extreme point of $DX = d$, $X \geqslant 0$), we start dealing with (II.2).

Find $X_{\binom{2}{1}}$ and $u_{\binom{2}{1}}$. As feasible region of (II.2) contains the feasible region of (II.1) and as $X_{\binom{1}{1}} \cap S = \phi$, $u_{\binom{2}{1}} \geqslant u_{\binom{1}{1}}$ it follows that $X_{\binom{2}{1}} \cap S = \phi$. Find values $R_i$ ($\geqslant u_{\binom{1}{2}}$) of the objective function at the extreme points adjacent to elements of $X_{\left(\frac{2}{1}\right)}$. Pick up that value (say $W_1$) which is nearest to $u_{\left(\frac{1}{2}\right)}$ i.e., $W_1 = \underset{i}{\text{Min}}\,[R_i]$. Find out the extreme points belonging to $S_2$ corresponding to $W_1$. Again find values ($< W_1$ but $\geqslant u_{\left(\frac{1}{2}\right)}$) at all extreme points adjacent to the elements of the set of extreme points yielding value $W_1$. Out of these choose that value (say $W_2$) which is nearest to $u_{\left(\frac{1}{2}\right)}$. Determine the set of extreme points ($\in S_2$) corresponding to the value $W_2$ and again find out the values ($< W_2$ but $\geqslant u_{\left(\frac{1}{2}\right)}$) of the objective function at extreme points adjacent to the elements of the set of extreme points yielding $W_2$. Select the value (say $W_3$) which is nearest to $u_{\left(\frac{1}{2}\right)}$. This is repeated as far as possible. Suppose we can go upto $W_j$ ($\geqslant u_{\left(\frac{1}{2}\right)}$ but $< W_{j-1}$). At this stage a cut (which we call as 'strong cut') $CX \leqslant W_j$ is introduced to the problem (II.2) and we generate a new problem :

$$\left.\begin{aligned} &\text{Max } CX \\ &\text{subject to } DX = d \\ &\qquad\qquad CX \leqslant W_j \\ &\qquad\qquad X \geqslant 0 \end{aligned}\right\} \quad \text{...Problem (II.3)}$$

Find the set $X_{\left(\frac{3}{1}\right)} = \left[X^3_{1\,1}, X^3_{1\,2}, \ldots, X^3_{1\,t_1}\right]$ of all the optimal extreme points of (II.3). Value of objective function at elements of $X_{\left(\frac{3}{1}\right)}$ will be $W_j$. As $W_j \geqslant u_{\left(\frac{1}{2}\right)}$ and as $X_{\left(\frac{1}{2}\right)} \cap S = \phi$, it follows that $X_{\left(\frac{3}{1}\right)} \cap S = \phi$ i.e., elements of $X_{\left(\frac{3}{1}\right)}$ are not feasible with respect to $AX = b$. Clearly $X_{\left(\frac{3}{1}\right)} \neq \phi$. Find the set $X_{\left(\frac{3}{2}\right)} = \left[X^3_{2\,1}, X^3_{2\,2}, \ldots\ldots, X^3_{2\,t_2}\right]$ of second best extreme point solutions of (II.3). If $X_{\left(\frac{3}{2}\right)} = \phi$, then (I) has no solution. If $X_{\left(\frac{3}{2}\right)} \neq \phi$ and $X_{\left(\frac{3}{2}\right)} \cap S \neq \phi$, then every $X$ belonging to $X_{\left(\frac{3}{2}\right)} \cap S$ will be an optimal solution of (I). But if $X_{\left(\frac{3}{2}\right)} \cap S = \phi$, find value $u_{\left(\frac{3}{2}\right)}$ of the objective function at elements of $X_{\left(\frac{3}{2}\right)}$ and introduce a cut $CX \leqslant u_{\left(\frac{3}{2}\right)}$ ($< W_j$) to the problem (II.2) and get the new problem as :

$$\left.\begin{aligned} &\text{Max } CX \\ &\text{subject to } DX = d \\ &\qquad\qquad CX \leqslant u_{\left(\frac{3}{2}\right)} \\ &\qquad\qquad X \geqslant 0. \end{aligned}\right\} \quad \text{...Problem (II.4)}$$

Process is repeated over this and the subsequent generated problems till we get $X^i_{(2)} \cap S \neq \phi$, $i \geqslant 4$ in which case every $X \in X^i_{(2)} \cap S$ will be an optimal solution of (I) where $X^i_{(2)}$ is the set of second best extreme point solutions of the problems :

$$\left.\begin{aligned} &\text{Max } CX \\ &\text{subject to } DX = d \\ &\qquad\qquad CX \leqslant u^{i-1}_{(2)} \\ &\qquad\qquad X \geqslant 0. \end{aligned}\right\} \quad i \geqslant 4 \quad \text{...Problem (II.}i\text{)}$$

and $u^{i-1}_{(2)}$ is the value of the objective function at second best extreme point solutions of the problem (II.$i-1$). It is quite clear that for each $i\,(\geqslant 4)$, $X^{i-1}_{(2)} \in S_2$ and $X^{i-1}_{(2)} \subseteq X^{i}_{(1)}$.

As cuts are nothing but parallel hyperplanes, a cut at any stage makes the preceding cuts redundant. The process will definitely converge as we move from one extreme point to another extreme point of $DX = d$, $X \geqslant 0$ and these extreme points are finite in number. This method of 'strong cut' may coincide with the method of 'deep cut' in some cases, but in general this new approach will move towards optimality at a much faster rate.

If the 'strong cut' passes through the $j$th best extreme point (for definition see Appendix) of $DX = d$, $X \geqslant 0$, then clearly we save ourselves from the trouble of determining 2nd, 3rd, ..., $j$th best extreme points of $DX = d$, $X \geqslant 0$ and jump direct to $j$th best extreme point.

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## Appendix

(i) *Definition of second best extreme point solution* (Hadley 1962, Kirby *et al.* 1972a, Puri and Swarup 1973) — If $X_{(\frac{1}{2})}$ is the set of second best extreme point solutions of (II.1), then $X_{(\frac{1}{2})} \in S_1 \backslash X_{(\frac{1}{1})}$

and $u_{(\frac{1}{2})} \geqslant CX$ for every $X \in S_1 \backslash X_{(\frac{1}{1})}$.

We can also say that $X_{(\frac{1}{2})}$ is the set of optimal extreme point solutions of the problem :

$$\text{Max } CX$$

$$X \in S_1 \backslash X_{(\frac{1}{1})}.$$

(ii) *Method of finding second best extreme point solutions* (Hadley 1962, Kirby *et al.* 1972a, Puri and Swarup 1973) — For explanation let us consider problem (II.1). Find $X_{(\frac{1}{1})}$. For each simplex tableau with basis $B$ corresponding to an element of $X_{(\frac{1}{1})}$ find the following :

$$H(B) = [j/z_j - c_j > 0]$$

$$= \text{set of non-basic variables having positive } z_j - c_j.$$

$$\underset{j \in H(B)}{\theta_j} = \underset{i}{\text{Min}} \left[ \frac{x_{B_i}}{y_{ij}},\ y_{ij} > 0 \right]$$

$$\gamma_B = \underset{j \in H(B)}{\text{Min}} \ [\theta_j(Z_j - C_j)/\theta_j > 0]$$

$$\delta = \text{Min}\ [\gamma_B/B \text{ is a basis for an element of } X_{(\frac{1}{1})}]$$

where $x_{B_i}$ is the $i$th component of the vector $X_B = B^{-1}f$ of basic variables, $y_{ij}$ is the $i$th component of the vector $Y_j = B^{-1}f_j$, $f_j$ being $j$th column of $F$, $z_j = C_B Y_j$ where $C_B$ is the vector of prices of basic variables.

If $\delta = \theta_k(Z_k - C_k) = \dfrac{x_{B_r}}{y_{rk}}(Z_k - C_k)$, then a simplex iteration bringing $f_k$ into the basis and departing column corresponding to basic variable $X_{B_r}$ is performed on the simplex tableau with basis $B$. This will give set of second best extreme points of (II.1). [When basis yielding $\delta$ is unique we get a unique second best extreme point, otherwise we get a set of second best extreme point].

(iii) *Definition of* j*th best extreme point* — Set $X^2_{(j)}$ of jth best extreme point solutions of (II.2) belongs to the set $S_2 \setminus \bigcup_{i=1}^{j-1} X^2_{(i)}$ and is such that

$$u^2_{(j)} \geqslant CX \text{ for all } X \in S_2 \setminus \bigcup_{i=1}^{j-1} X^2_{(i)}$$

where $u^2_{(j)}$ is the value of the objective function at elements of $X^2_{(j)}$

*Example :* Max $x_1 + 8x_2$

subject to
$$-7x_1 + 2x_2 \leqslant 4$$
$$9x_1 + 10x_2 \leqslant 64$$

and $(x_1, x_2)$ is an extreme point of

$$-x_1 + 2x_2 \leqslant 10$$
$$x_1 + 2x_2 \leqslant 14$$
$$2x_1 + x_2 \leqslant 16$$
$$x_1 - x_2 \leqslant 5$$
$$x_1, x_2 \geqslant 0.$$

*Solution :* Problem (II.1) of theory is :

$$\left.\begin{array}{l} \text{Max } Z = x_1 + 8x_2 \\ \text{subject to } \left.\begin{array}{r} -7x_1 + 2x_2 + x_3 = 4 \\ 9x_1 + 10x_2 + x_4 = 64 \end{array}\right] \equiv [AX = b \\ \qquad\qquad \left.\begin{array}{r} -x_1 + 2x_2 + x_5 = 10 \\ x_1 + 2x_2 + x_6 = 14 \\ 2x_1 + x_2 + x_7 = 16 \\ x_1 - x_2 + x_8 = 5 \end{array}\right] \equiv [DX = d \\ \qquad\qquad x_1, x_2, \ldots, x_8 \geqslant 0 \end{array}\right\} \quad \ldots\text{Problem (II.1)}$$

$D$ is $4 \times 8$. Therefore, $p = 4$.

*Step* 1 — $X_{\binom{1}{1}} = [X^1_{11} = (1, 11/2, 0, 0, 0, 2, 17/2, 19/2)]$

Number of non-null columns of $D$ corresponding to non-zero basic variables in $X^1_{11} = 5 > p$.

$\therefore X_{\binom{1}{1}} \cap S = \phi$

$u_{\binom{1}{1}} = 45.$

*Step* 2 — $X_{\binom{1}{2}} = [X^1_{21} = (0, 2, 0, 44, 6, 10, 14, 7)]$.

Number of non-null columns of $D$ corresponding to non-zero basic variables in $X^1_{21} = 5 > p$.

$\therefore X_{\binom{1}{2}} \cap S = \phi$

$u_{\binom{1}{2}} = 16.$

Now we start dealing with problem (II.2) of theory which is :

$$\left.\begin{array}{l} \text{Max } x_1 + 8x_2 \\ \text{subject to } \begin{array}{r} -x_1 + 2x_2 + x_5 = 10 \\ x_1 + 2x_2 + x_6 = 14 \\ 2x_1 + x_2 + x_7 = 16 \\ x_1 - x_2 + x_8 = 5 \\ x_1, x_2, \ldots, x_8 \geqslant 0 \end{array} \end{array}\right\} \quad \ldots\text{Problem (II.2)}$$

*Step* 3 — $X_{\binom{2}{1}} = [X^2_{11} = (2, 6, 0, 0, 0, 0, 6, 9)]$.

As $X^2_{11}$ is not feasible with respect to $AX = b$, we have

$X_{\binom{2}{1}} \cap S = \phi, \; u_{\binom{2}{1}} = 50.$

*Step* 4 — Values $R_i$ of the objective function at extreme points of (II.2) adjacent to $X^2_{11}$ are :

$R_1$ (when $d_6$ enters and $d_1$ leaves) $= 40 > u_{\binom{1}{2}}$

$R_2$ (when $d_5$ enters and $d_7$ leaves) $= 38 > u_{\binom{1}{2}}$

where $d_1, d_2, \ldots, d_8$ are columns of $D$.

$W_1 = \min_i [R_i] = 38.$

Extreme point of (II.2) corresponding to $W_1$ is (6, 4, 0, 0, 8, 0, 0, 3) value ($< w_1$ but $\geqslant u_{(\frac{1}{2})}$) of objective function at extreme point adjacent to (6, 4, 0, 0, 8, 0, 0, 3) is = 23.

$$W_2 = 23 > u_{(\frac{1}{2})}.$$

Extreme point of (II.2) yielding value 23 is (7, 2, 0, 0, 13, 3, 0, 0) which is obtained from basis of the extreme point solution (6, 4, 0, 0, 8, 0, 0, 3) by entering $d_6$ and departing $d_8$.

Value ($< W_2$) of the objective function at extreme point adjacent to (7, 2, 0, 0, 13, 3, 0, 0) is 5 which is less than $u_{(\frac{1}{2})}$.

So we will go upto $W_2$ only.

Thus here, $W_j = W_2 = 23$.

Introduce a cut $x_1 + 8x_2 \leqslant 23$ to the problem (II.2). New generated problem is :

$$\left.\begin{array}{l} \text{Max } Z = x_1 + 8x_2 \\ \text{subject to} \quad \left.\begin{array}{r} -x_1 + 2x_2 + x_5 = 10 \\ x_1 + 2x_2 + x_6 = 14 \\ 2x_1 + x_2 + x_7 = 16 \\ x_1 - x_2 + x_8 = 5 \\ x_1 + 8x_2 + x_9 = 23 \end{array}\right] \equiv [GX = g \text{ (say)} \\ \qquad\qquad x_1, x_2, \ldots, x_9 \geqslant 0 \end{array}\right\} \quad \text{...Problem (II.3)}$$

*Step* 5 — $X_{(\frac{3}{1})} = [X^3_{1\,1} = (7, 2, 0, 0, 13, 3, 0, 0, 0),$
$X^3_{1\,2} = (0, 23/8, 0, 0, 17/4, 33/4, 105/4, 63/5, 0)]$

$X_{(\frac{3}{1})} \cap S = \phi.$

Simplex tableau for $X^3_{1\,1}$ is :

| | | $C_j \rightarrow$ | 1 | 8 | 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Vectors in basis | $X_B$ | $g_1$ | $g_2$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ |
| 0 | $g_5$ | 13 | 0 | 0 | 1 | 0 | 0 | 10/9 | $-\frac{1}{9}$ |
| 0 | $g_6$ | 3 | 0 | 0 | 0 | 1 | 0 | $-\frac{6}{9}$ | $-\frac{1}{3}$ |
| 0 | $g_7$ | 0 | 0 | 0 | 0 | 0 | 1 | $-15/9$ | $-\frac{1}{3}$ |
| 1 | $g_1$ | 7 | 1 | 0 | 0 | 0 | 0 | $\frac{8}{9}$ | $\frac{1}{9}$ |
| 8 | $g_2$ | 2 | 0 | 1 | 0 | 0 | 0 | $-\frac{1}{9}$ | $\frac{1}{9}$ |
| | $Z_j - C_j \rightarrow$ | $Z=23$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

$g_1, g_2, \ldots, g_9$ are columns of $G$; $g_3, g_4$ are null columns.

Simplex tableau for $X^3_{12}$ is :

| | | $C_j \rightarrow$ | 1 | 8 | 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Vectors in basis | $X_B$ | $g_1$ | $g_2$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ |
| 0 | $g_5$ | 17/4 | $-\frac{5}{4}$ | 0 | 1 | 0 | 0 | 0 | $-\frac{1}{4}$ |
| 0 | $g_6$ | 33/4 | $\frac{3}{4}$ | 0 | 0 | 1 | 0 | 0 | $-\frac{1}{4}$ |
| 0 | $g_7$ | 105/4 | 15/8 | 0 | 0 | 0 | 1 | 0 | $-\frac{1}{8}$ |
| 0 | $g_8$ | 63/8 | $\frac{9}{8}$ | 0 | 0 | 0 | 0 | 1 | $\frac{1}{8}$ |
| 8 | $g_2$ | 23/8 | $\frac{1}{8}$ | 1 | 0 | 0 | 0 | 0 | $\frac{1}{8}$ |
| | $Z_j - C_j \rightarrow$ | $Z=23$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

$B^3_{11}$ = basis for $X^3_{11}$ = $(g_5, g_6, g_7, g_1, g_2)$

$B^3_{12}$ = basis for $X^3_{12}$ = $(g_5, g_6, g_7, g_8, g_2)$

$H_{(B^3_{11})} = 9$

$$\theta_9 = \text{Min}\left[\frac{7}{\frac{1}{9}}, \frac{2}{\frac{1}{9}}\right] = 18$$

(for $B^3_{11}$)

$\gamma_{(B^3_{11})} = \text{Min}\,[18 \times 1] = 18$

$H_{(B^3_{12})} = 9$

$$\theta_9 = \text{Min}\left[\frac{63/8}{\frac{1}{8}}, \frac{23/8}{\frac{1}{8}}\right] = 23$$

(for $B^3_{12}$)

$\gamma_{(B^3_{12})} = \text{Min}\,[23 \times 1] = 23$

$$\delta = \text{Min}\left[\gamma_{B^3_{11}}, \gamma_{B^3_{12}}\right] = \text{Min}\,[18, 23] = 18.$$

Therefore, second best extreme point $X^3_{21}$ of (II.3) will be obtained from simplex tableau of $X^3_{11}$ by entering $g_9$ and departing $g_2$. Simplex tableau for $X^3_{21}$ is :

| | | $C_j \rightarrow$ | 1 | 8 | 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Vectors in basis | $X_B$ | $g_1$ | $g_2$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ |
| 0 | $g_5$ | 15 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 0 | $g_6$ | 9 | 0 | 3 | 0 | 1 | 0 | $-1$ | 0 |
| 0 | $g_7$ | 6 | 0 | 3 | 0 | 0 | 1 | $-2$ | 0 |
| 1 | $g_1$ | 5 | 1 | $-1$ | 0 | 0 | 0 | 1 | 0 |
| 0 | $g_9$ | 18 | 0 | 9 | 0 | 0 | 0 | $-1$ | 1 |
| | $Z_j - C_j \rightarrow$ | $Z=5$ | 0 | $-9$ | 0 | 0 | 0 | 1 | 0 |

$X_{\binom{3}{2}} = [X^3_{2\,1} = (5, 0, 0, 0, 15, 9, 6, 0, 18)]$

Clearly $X^3_{2\,1}$ satisfies feasibility in $AX = b$.

Therefore, $X^3_{2\,1} \cap S \neq \phi$ and hence $X^3_{2\,1}$ will be the required solution.

Therefore, optimal solution of the original problem is

$$x_1 = 5,\ x_2 = 0$$

and optimal value is 5.