DECOMPOSITIONS IN TOPOLOGICAL VECTOR SPACES

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This paper deals with the most natural question in the decomposition theory i.e., 'Under what circumstances does a given sequence of subspaces of a Topological Vector Space constitute a Schauder decomposition?' The answer given in section 2 is, in fact, a generalized version of a result obtained by McArthur and Retherford (1965, 1966) for a Banach space. As a consequence of the above result we study an intrinsic property of a Schauder decomposition in a metrizable topological vector space.

1. INTRODUCTION

Let E be a topological vector space (TVS) whose topology is given by a family $\{p_\lambda : \lambda \in D\}$ of pseudo-norms (Hyers 1939, p. 630), where D is a directed set.

A sequence $\{M_n\}_{n=1}^{\infty}$ of non-trivial subspaces of E is said to be a decomposition of E, if for each $x \in E$ there is a unique sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n \in M_n$ for each n $\geqslant 1$ of vectors in E, such that $x = \lim_n \sum_{i \leqslant n} x_i$.

Remark : The notion of the decompositions is a generalization of the notion of bases in TVS.

The decomposition $\{M_n\}_{n=1}^{\infty}$ induces a sequence $\{P_n\}_{n=1}^{\infty}$ of projections defined by $P_n(x) = x_n$, where $x = \lim_n \sum_{i \leqslant n} x_i$. These projections are orthogonal i.e. $P_m P_n = 0$ if $m \neq n$ and $P_n(E) = M_n$. $\{P_n\}_{n=1}^{\infty}$ are known as orthogonal projections associated with $\{M_n\}_{n=1}^{\infty}$. We may define another sequence of projections Q_n as follows : $Q_n = \sum_{i \leqslant n} P_i$ (i.e. $Q_n(x) = \sum_{i \leqslant n} P_i(x)$) and then Q_n's satisfy $Q_m Q_n = Q_{min\,(m,\,n)}$. Thus if $\{M_n\}_{n=1}^{\infty}$ is a decomposition for E, then we may express each $x \in E$ either as $x = \lim_n \sum_{i \leqslant n} P_i(x)$ or as $x = \lim_n Q_n(x)$.

A decomposition of E is said to be a Schauder decomposition if the corresponding orthogonal projections P_n are continuous.

A decomposition of E is said to be an equi-Schauder (or e-Schauder) decomposition (see McArthur and Retherford 1965, p. 207) if the corresponding sequence $\{Q_n\}_{n=1}^{\infty}$ of projections is equicontinuous. Since continuity of each Q_n $(n \geqslant 1)$ implies and is implied by the continuity of each P_n it follows that an equi-Schauder decomposition is a Schauder decomposition.

Throughout the rest of this paper we will write that $\{M_n\}_{n=1}^{\infty}$ is a decomposition for a topological vector space E and $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ are sequences of projections associated with this decomposition defined as above. $\{p_\lambda : \lambda \in D\}$; D being a directed set; will represent throughout the family of pseudo-norms defining the topology of E.

In §2 we will give characterization of a Schauder decomposition and in §3 we prove a result which is the application of results in §2.

2. Characterization of Schauder Decomposition

The following proposition is an immediate consequence of the definition.

Proposition 2.1 — Let E be a Hausdorff topological vector space, $\{P_n\}_{n=1}^{\infty}$ be a sequence of continuous projections on E satisfying the following conditions :

(i) $P_i P_j = \delta_{ij} P_i$

(ii) $x = \lim\limits_{n} \sum\limits_{i \leqslant n} P_i(x)$, for each $x \in E$

then there exists a Schauder decomposition of E whose associated sequence of projections is given by $\{P_n\}_{n=1}^{\infty}$.

Corollary 2.1 — Let E be as above and $\{Q_n\}_{n=1}^{\infty}$ be a family of continuous projections on E satisfying

(i) $Q_i Q_j = Q_{min\,(i,j)}$

(ii) $x = \lim\limits_{n} Q_n(x)$ for each $x \in E$

then there exists a Schauder decomposition of E.

Proof : Let $P_1 = Q_1$, $P_i = Q_i - Q_{i-1}$, $i \geqslant 2$. Then $\{P_n\}_{n=1}^{\infty}$ form a sequence of continuous projections satisfying (i) and (ii) of Proposition 2.1. Hence the result follows from the Proposition 2.1.

Definition 2.1 — A family H of linear maps from a topological vector space E into a topological vector space F is said to be an equicontinuous family (Horváth 1966, p. 198) if for each neighbourhood V of $0 \in F$ there exists a neighbourhood U of $0 \in E$ such that whenever $x \in U$ then $h(x) \in V$ for all $h \in H$.

Let $\{P_\lambda : \lambda \in D\}$ and $\{q_i : i \in D'\}$ be the families of pseudonorms which generate the topologies of E and F respectively. Then H is an equicontinuous family of linear maps from E into F if and only if for each $i \in D'$ there exists a constant $K_i > 0$ and a $\lambda \in D$ such that

$$q_i\,[h(x)] \leqslant K_i\, P_\lambda(x) \qquad \ldots(1)$$

for all $x \in E$ and $h \in H$.

The proof of the following proposition follows from the above characterization of an equicontinuous family of linear maps.

Proposition 2.2 — Let E be a Hausdorff topological vector space. Let $\{M_n\}_{n=1}^{\infty}$ be an equi-Schauder decomposition of E then for each $\lambda \in D$ there is a constant $K_\lambda \geqslant 1$ (suffix λ shows that K_λ is dependent on λ) and a $\mu \in D$ such that

$$p_\lambda\Big(\sum_{i=1}^{m} x_i\Big) \leqslant K_\lambda\, p_\mu\Big(\sum_{i=1}^{n} x_i\Big)$$

for all integers m, n with $n \geqslant m$ and for any sequence $\{x_i\}$ of E where $x_i \in M_i$.

In the following theorem we give sufficient conditions for a sequence of non-trivial closed subspaces in a topological vector space to become a Schauder decomposition. This theorem in the case of bases in a Banach space is due to Nikolskii (Marti 1969 Th. 7, p. 93 and Singer 1970, p. 58)

Theorem 2.1 — Let E be a complete Hausdorff topological vector space. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of non-trivial closed subspaces of E such that $\overline{Sp} \bigcup_{n=1}^{\infty} M_n = E$. Then $\{M_n\}_{n=1}^{\infty}$ is a Schauder decomposition of E, if for every $\lambda \in D$ there exists a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda\Big(\sum_{i=1}^{m} x_i\Big) \leqslant K_\lambda\, p_\mu\Big(\sum_{i=1}^{n} x_i\Big),$$

for all integers $m, n \geqslant 1$, where $n \geqslant m$ and for all sequences $\{x_i\}$ of E with $x_i \in M_i$.

PROOF : Let $E_n = \overline{Sp} \bigcup_{i=1}^{n} M_i$. Then E_n is a closed subspace of E. Starting from $E_1 = M_1$ we show by induction that $E_n = \bigoplus_{i=1}^{n} M_i$ for each $n \geqslant 1$, i.e. we prove that $E_n \cap M_{n+1} = \{0\}$. Suppose $E_n = \bigoplus_{i=1}^{n} M_i$ for some $n \geqslant 1$ and let $x \in E_n \cap M_{n+1}$ be arbitrary $\Rightarrow x \in E_n \Rightarrow x = \sum_{i=1}^{n} x_i$, $x_i \in M_i$ $(1 \leqslant i \leqslant n)$. Again $x \in M_{n+1}$. Let $\lambda \in D$ be arbitrary, then by the given condition

$$p_\lambda(x) = p_\lambda(\sum_{i=1}^{n} x_i) \leqslant K_\lambda\, p_\mu(\sum_{i=1}^{n+1} x_i), \text{ for some } \mu \in D,$$

where $x_{n+1} = -x$. Hence $p_\lambda(x) = 0$ and as $\lambda \in D$ is arbitrary then $x = 0$ (since E is Hausdorff). Hence $E_{n+1} = \bigoplus_{i=1}^{n+1} M_i$.

For n $> m$, we define $Q_{mn} : E_n \to E_m$ by $Q_{mn}(z) = x$

where $z \in E_n \Rightarrow z = x + y$, $x \in E_m$, $y \in M_{m+1} \oplus M_{m+2} \oplus M_{m+3} \oplus \ldots \oplus M_n$. If $n \leqslant m$ we take Q_{mn} the natural embedding of E_n in E_m. By the uniqueness of the decomposition of $z \in E_n$, it follows that Q_{mn} is well-defined and also it is a projection of E_n onto E_m.

Now from the hypothesis it is evident that for any $\lambda \in D$ there is a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda[Q_{mn}(z)] = p_\lambda(x) \leqslant K_\lambda\, p_\mu(x + y) = K_\lambda\, p_\mu(z),$$

for all $z \in E_n$. Hence Q_{mn} is continuous for $m, n \geqslant 1$. Let now

$$F = \{x \in E : x \in E_n \text{ for some } n < \infty\}. \text{ Clearly } \mathrm{Sp} \bigcup_{n=1}^{\infty} M_n \subset F.$$

Hence $\overline{F} = E$. Define $Q'_m : F \to E_m$ by $Q'_m(y) = Q_{mn}(y)$, where $y \in E_n$ for some $n < \infty$. Since each Q_{mn} is continuous for $m, n \geqslant 1$, therefore Q'_m is also continuous.

Moreover, $Q'^2_m(y) = Q_{mm}[Q'_m(y)] = Q'_m(y) \Rightarrow Q'_m$ is a projection. Hence Q'_m has a unique continuous extension Q_m on $\overline{F} = E$ (Horváth 1966, p. 129).

Since E_m is closed the range of Q_m is E_m. Now we shall show that $\{Q_n\}_{n=1}^{\infty}$ is equicontinuous. For, let $x \in E$ then there exists a net $\{x^{(\alpha)}\}_{\alpha \in \Delta}$ of F such that $x^{(\alpha)} \to x \Rightarrow p_\lambda(x^{(\alpha)} - x) \to 0$ for each $\lambda \in D$ and $Q_m(x^{(\alpha)}) \to Q_m(x)$. Let $\lambda \in D$ be arbitrary and m be chosen arbitrarily but fixed, then

$$p_\lambda[Q_m(x)] = p_\lambda[Q_m(x-x^{(\alpha)}) + Q_m(x^{(\alpha)})]$$
$$\leqslant p_\rho\,[Q_m(x-x^{(\alpha)})] + K_\rho\, p_\eta\,(x^{(\alpha)}), \text{ for some } \rho \text{ and } \eta \in D;$$

by the given condition and the definition of Q_m it follows that η is dependent only on ρ but not on m. Hence

$$p_\lambda[Q_m(x)] \leqslant p_\rho\,[Q_m(x-x^{(\alpha)})] + K_\rho\, p_\eta(x^{(\alpha)}-x)$$
$$\leqslant p_\rho\,[Q_m(x-x^{(\alpha)})] + K_\rho\, p_\mu(x^{(\alpha)}-x) + K_\rho\, p_\mu(x),$$
$$\text{for some } \mu \in D. \quad \ldots(2)$$

But $Q_m(x^{(\alpha)}-x) \to 0$ (m fixed) hence $p_\rho\,[Q_m(x^{(\alpha)}-x)] \to 0$, for all $\rho \in D$. So by (2) we have $p_\lambda[Q_m(x)] \leqslant K_\rho\, p_\mu(x)$. Since in the preceding analysis, ρ depends on λ and K_ρ depends on ρ, hence K_ρ depends on λ. Thus we can write $K_\rho \equiv K_\lambda$. Consequently

$$p_\lambda[Q_m(x)] \leqslant K_\lambda\, p_\mu(x), \text{ for all } x \in E. \quad \ldots(3)$$

Hence $\{Q_m\}_{m=1}^{\infty}$ is equicontinuous.

Next, we show that $x = \lim\limits_{n} Q_n(x)$ for each $x \in E$. For, let $x \in E$, $\lambda \in D$ be arbitrary but fixed and $\epsilon > 0$ be given. Then

$$P_\lambda\,[Q_n(x)-x] \leqslant p_\mu[Q_n(x)-y] + p_\mu(x-y), \text{ for some } \mu \in D \text{ and for all } y \in E.$$

Now corresponding to $\mu \in D$ there is a $\eta \in D$ satisfying (3). Again we can find $\rho \in D$ such that

$$p_\rho\,(x) \geqslant \max\,[p_\mu(x), p_\mu(x)] \text{ for all } x \in E.$$

As $\bar{F} = E$, there is, therefore a $m < \infty$ corresponding to $\rho \in D$ and $\epsilon > 0$, such that $p_\rho\,(x-y) < \epsilon$, for some $y \in E_m$. Hence with this choice of y, one gets

$$p_\lambda\,[Q_n(x)-x] \leqslant p_\mu[Q_n(x-y)] + p_\mu(x-y), \text{ for } n \geqslant m,\ y \in E_m$$
$$\leqslant (K_\mu + 1)\, p_\rho\,(x-y), \text{ for } n \geqslant m,\ y \in E_m$$
$$< (K_\mu + 1)\,\epsilon, \text{ for } n \geqslant m.$$

Since p_λ is arbitrary, it follows that $\lim\limits_{n}\ Q_n(x) = x$. Again $Q_m Q_n = Q_{min\,(m,n)}$ and $Q_1(E) = M_1$, $(Q_i - Q_{i-1})(E) = M_i$ for $i \geqslant 2$, and so by Corollary 2.1, the result follows.

Remark : In fact, the above decomposition is an equi-Schauder decomposition as the projection sequence $\{Q_n\}_{n=1}^{\infty}$ is equicontinuous.

Thus Proposition 2.2 together with the Theorem 2.1 can be restated as follows.

Theorem 2.2 — Let E be a complete Hausdorff topological vector space and $\{M_n\}_{n=1}^{\infty}$ be a sequence of non-trivial closed subspaces of E such that $\overline{Sp} \bigcup_{i=1}^{\infty} M_i = E$. Then $\{M_n\}_{n=1}^{\infty}$ is an equi-Schauder decomposition if and only if for each $\lambda \in D$ there exists a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda \left(\sum_{i=1}^{m} x_i \right) \leqslant K_\lambda \, p_\mu \left(\sum_{i=1}^{n} x_i \right)$$

for all integers m, n where $n \geqslant m$ and for all sequences $\{x_i\}$ of E with $x_i \in M_i$.

The following theorem gives a necessary condition for a Schauder decomposition of a topological vector space having a certain property.

Theorem 2.3 — Let E be a Hausdorff topological vector space containing a set of second category. If a sequence $\{M_n\}_{n=1}^{\infty}$ of non-trivial closed subspaces of E is a Schauder decomposition for E, then for each $\lambda \in D$, there is a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda \left(\sum_{i=1}^{m} x_i \right) \leqslant K_\lambda \, p_\mu \left(\sum_{i=1}^{n} x_i \right),$$

for all integers m, n with $n \geqslant m$, and for all sequences $\{x_i\}$ of E with $x_i \in M_i$.

PROOF: Since $\{M_n\}_{n=1}^{\infty}$ is a Schauder decomposition then the projections $\{Q_n\}_{n=1}^{\infty}$ are continuous.

Since for each $x \in E$, $x = \lim_n Q_n(x)$, therefore $\{Q_n\}$ is pointwise bounded on E and hence on the set of second category which is contained in E.

Therefore, $\{Q_n\}_{n=1}^{\infty}$ is equicontinuous (Kelley and Namioka 1963, p. 104). Thus given $\lambda \in D$, there is a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda[Q_m(x)] \leqslant K_\lambda \, p_\mu(x) \qquad \ldots(4)$$

for all $x \in E$ and all $m \geqslant 1$. Let m, n be two integers with $n \geqslant m$. If $x = \sum_{i=1}^{n} x_i$, where $x_i \in M_i$, then from (4) we get

$$p_\lambda \left(\sum_{i=1}^{m} x_i \right) \leqslant K_\lambda \, p_\mu \left(\sum_{i=1}^{n} x_i \right).$$

Hence the result.

Theorem 2.1 and Theorem 2.3 can be put together in the following form.

Theorem 2.4* — Let E be a complete Hausdorff topological vector space containing a set of second category. A sequence $\{M_n\}_{n=1}^{\infty}$ of non-trivial closed subspace of E such that $\overline{Sp} \bigcup_{i=1}^{\infty} M_i = E$ is a Schauder decomposition of E if and only if for each $\lambda \in D$ there exists a $\mu \in D$ and a constant $K_\lambda \geqslant 1$ such that

$$p_\lambda(\sum_{i=1}^{m} x_i) \leqslant K_\lambda \, p_\mu(\sum_{i=1}^{n} x_i)$$

for all integers m, n with $n \geqslant m$ and for all sequence $\{x_i\}$ of E with $x_i \in M_i$.

Definition 2.2 — A topological vector space E is metrizable if there exists a metric on E such that the topology defined by it coincides with the linear topology of E. By Theorem 1 of Horváth (1966, p. 111) it follows that the topology of a metrizable topological vector space is generated by a translation invariant metric d (Horváth 1966, p. 110) such that the distance $|x| = d(x, 0)$ from the origin satisfies :

(a) $|x| = 0$ iff $x = 0$;
(b) $|x+y| \leqslant |x| + |y|$, for $x, y \in E$;
(c) $|\lambda| \leqslant 1 \Rightarrow |\lambda x| \leqslant |x|$, equality holds when $|\lambda| = 1$;
(d) $\lambda \to 0 \Rightarrow |\lambda x| \to 0$ for every $x \in E$.

A complete metrizable topological vector space is known as a F-space and the above distance function $|\ldots|$ satisfying (a) – (d) is a F-norm.

As a particular case of Theorem 2.4, in the case of a F-space we get the following.

Theorem 2.5 — Let E be a F-space. A sequence $\{M_n\}_{n=1}^{\infty}$ of non-trivial closed subspaces of E such that $\overline{Sp} \bigcup_{i=1}^{\infty} M_i = E$ is a Schauder decomposition of E if and only if there exists a constant $K \geqslant 1$ such that

$$|\sum_{i=1}^{m} x_i| \leqslant K |\sum_{i=1}^{n} x_i|$$

for all integers m, n with $n \geqslant m$ and for all sequences $\{x_i\}$ of E with $x_i \in M_i$.

*This theorem generalizes a result proved by Kamthan and Gupta : *Tamkang J. Math.*, 7 (1976), 51-55.

3. Applications

In the discussion of this section E will always represent, unless otherwise specified, a complete Hausdorff topological vector space containing a set of second category. It is easy to see that if a sequence $\{M_n\}_{n=1}^{\infty}$ of subspaces of E is a Schauder decomposition for $[M_n]$ (the closed linear span of $\bigcup_{n=1}^{\infty} M_n$) then any sequence $\{x_n\}$ of E with $0 \neq x_n \in M_n$ for $n \geqslant 1$ is basic (see Singer 1970, p. 27). The converse of this statement is not true (Davis 1968, p. 1084). However, we do have a partial converse for F-space in the form of the following theorem.

Theorem 3.1 — Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of non-trivial closed subspaces of a F-space E such that each sequence $\{x_n\} \subset E$ with $0 \neq x_n \in M_n$ is basic. Then there exists an integer N such that $\{M_n\}_{n=N}^{\infty}$ is a Schauder decomposition of $[M_n : n \geqslant N]$ (i.e. the closed linear span of $\bigcup_{n=N}^{\infty} M_n$).

Let us first introduce a few notations. Suppose $u = \{x_n\}$ is a sequence in a TVS E with $0 \neq x_n \in M_n$ for $n \geqslant 1$ and let $u_m = \{x_n\}_{n=m}^{\infty}$. For some given $\lambda \in D$, let $K_\lambda(u_m)$ be the smallest constant such that

$$p_\lambda \left(\sum_{i=m}^{s} \alpha_i x_i \right) \leqslant K_\lambda \, p_\mu \left(\sum_{i=m}^{s+t} \alpha_i x_i \right)$$

holds for some $\mu \in D$, for all $K_\lambda \geqslant K_\lambda(u_m)$ and for all sequences $\{\alpha_i\}$ of scalars and integers $s \geqslant m$, t being an integer greater than or equal to 1. In the case of a F-space E, $K_\lambda(u_m)$, K_λ defined above will be replaced by $K(u_m)$, K respectively.

We now prove a lemma.

Lemma 3.1 — Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of non-trivial closed subspaces of E such that each $u = \{x_n\}$ with $0 \neq x_n \in M_n$ is basic. Then for each given $\lambda \in D$ there exists a positive integer N and a constant $K_\lambda \geqslant 1$ such that every sequence u as above has $K_\lambda(u_N) \leqslant K_\lambda$.

Proof : Let the condition be not true i.e. for a given $\lambda \in D$ there is no such N and K_λ satisfying the condition. Let $N = 1$, $K_\lambda = 2$, then there is $u^{(1)} = \{x_n^{(1)}\}_{n=1}^{\infty}$ with $0 \neq x_n^{(1)} \in M_n$ such that $K_\lambda(u^{(1)}) > 2$. Then there exist integers s_1; t_1 such that $s_1 < t_1$ and a finite sequence of scalars $\alpha_1, \alpha_2, \ldots \alpha_{t_1}$ such that

$$p_\lambda \left(\sum_{j=1}^{s_1} \alpha_j x_j^{(1)} \right) > 2 p_\mu \left(\sum_{j=1}^{t_1} \alpha_j x_j^{(1)} \right), \text{ for all } \mu \in D.$$

Next, let $N = t_1 + 1$ and $K_\lambda = 2^2$, then there is a sequence $u^{(2)} = \{x_n^{(2)}\}_{n=1}^{\infty}$ with $0 \neq x_n^{(2)} \in M_n$ such that $K_\lambda (u^{(2)}_{t_1+1}) > 2^2$, i.e. there exist integers s_2, t_2 such that $t_1 < s_2 < t_2$ and a finite sequence of scalars $\alpha_{t_1+1}, \alpha_{t_1+2}, \ldots \alpha_{t_2}$ such that

$$p_\lambda \left(\sum_{j=t_1+1}^{s_2} \alpha_j x_j^{(2)} \right) > 2^2 p_\mu \left(\sum_{j=t_1+1}^{t_2} \alpha_j x_j^{(2)} \right), \text{ for all } \mu \in D,$$

and so, in general, we get a sequence $u^{(k)} = \{x_n^{(k)}\}_{n=1}^{\infty}$ with $0 \neq x_n^{(k)} \in M_n$ and integers s_k, t_k such that $t_{k-1} < s_k < t_k$ where $k \geqslant 1$, $t_0 = 0$, satisfying

$$p_\lambda \left(\sum_{j=t_{k-1}+1}^{s_k} \alpha_j x_j^{(k)} \right) > 2^k p_\mu \left(\sum_{j=t_{k-1}+1}^{t_k} \alpha_j x_j^{(k)} \right), \text{ for all } \mu \in D.$$

Let us now choose the sequence $u = \{x_n\}_{n=1}^{\infty}$ where $x_n = x_n^{(k)}$, if $t_{k-1} < n \leqslant t_k$, $k \geqslant 1$, $t_0 = 0$. Clearly $0 \neq x_n \in M_n$. But by the above analysis u is not basic. This contradicts our hypothesis. Hence the lemma is proved.

In particular if we consider E to be a F-space, then the statement of the above lemma runs as follows :

Lemma 3.2 — Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of non-trivial closed subspaces of a F-space E such that each $u = \{x_n\}$ with $0 \neq x_n \in M_n$ is basic. Then there exists an integer $N \geqslant 1$ and a constant $K \geqslant 1$ such that every sequence u as above has $K(u_N) \leqslant K$.

Proof of Theorem 3.1 : Let $\{x_n\}_{n=N}^{\infty}$ be as sequence in E where $x_n \in M_n$ for $n = N, N+1, \ldots$ Clearly,

$$\left| \left(\sum_{n=N}^{N+s} x_n \right) \right| \leqslant K \left| \left(\sum_{n=N}^{N+t} x_n \right) \right|$$

where K is obtained in Lemma 3.2, for all integers s, t with $s < t$. Hence by Theorem 2.5 the result follows.

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