ON AFFINE CONNEXIONS IN ALMOST COMPLEX MANIFOLD

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(Communicated by R. S. Mishra, F.N.A.)

(*Received* 16 *January* 1974)

The purpose of the present paper is to define and study an affine connexion in almost complex manifold in terms of another restricted affine connexion such that the torsion tensors of the connexion are equal but opposite in sign.

1. INTRODUCTION

Let us consider an almost complex manifold M_{2n} of class C^{∞} such that

$$\bar{\bar{X}} = -X \qquad \text{...(1.1)}$$

for arbitrary vector field X, where $\bar{X} \overset{def}{=} F(X)$ and F is an almost complex structure to M_{2n}.

In an almost complex manifold a connexion D is called an F-connexion, M-connexion and $*O$-connexion if (Mishra and Ram Hit 1972)

$$(D_X F)(Y) = 0 \qquad \text{...(1.2)}$$

$$(D_X F)(Y) + (D_Y F)(X) = 0 \qquad \text{...(1.3)}$$

and

$$(D_X F)(Y) + (D_{\bar{X}} F)(\bar{Y}) = 0 \qquad \text{...(1.4)}$$

respectively.

Let S be the torsion tensor of an affine connexion D. The connexion D is called half symmetric, semi-symmetric and almost symmetric if

$$S(X, Y) - S(\bar{X}, \bar{Y}) = \overline{S(X, \bar{Y})} - \overline{S(\bar{X}, Y)} \qquad \text{...(1.5)}$$

$$n\, S(X, Y) = X\, T(Y) - Y\, T(X) + \bar{X}\, T(\bar{Y}) - \bar{Y}\, T(\bar{X}) \qquad \text{...(1.6)}$$

and

$$n\, S(X, Y) = X\, U(Y) - Y\, U(X) + \bar{X}\, U(\bar{Y}) - \bar{Y}\, U(\bar{X}) \qquad \text{...(1.7)}$$

respectively, where $T(X) \overset{def}{=} (C_1^1\ S)(X)$ and $U(X) \overset{def}{=} (C_1^1\ \bar{S})(X)$ and C_1^1 stands for contraction with first slot of S.

2. Affine Connexion

Let us define another affine connexion by the relation

$$B_X\ Y \overset{def}{=} -\ D_X\ Y + [X,\ Y] \qquad ...(2.1)$$

such that the torsion tensor of connexion B is $-S$.

Remark 2.1 : Since the torsions D and B are equal and opposite to each other, therefore, if D is half symmetric, semi-symmetric and almost symmetric then B is also half symmetric, semi-symmetric and almost symmetric respectively.

Theorem 2.1 — Given that D is an F-connexion, the condition for the connexion B to be an F-connexion is

$$[X,\ \bar{Y}] = \overline{[X,\ Y]}. \qquad ...(2.2)$$

Proof : Barring Y in (2.1) and using (1.1) and (1.2), we get

$$(B_X\ F)(Y) + \overline{B_X\ Y} = -\ \overline{D_X\ Y} + [X,\ \bar{Y}],$$

which on the use of (2.1) yields

$$(B_X\ F)(Y) = [X,\ \bar{Y}] - \overline{[X,\ Y]}.$$

If B is an F-connexion, we have (2.2).

Theorem 2.2 — Given that D is an M-connexion, the condition for connexion B to be an M-connexion is

$$[\bar{X},\ Y] = [X,\ \bar{Y}]. \qquad ...(2.3)$$

Proof : Barring Y in (2.1) and using (1.1), we get

$$(B_X\ F)(Y) = [X,\ \bar{Y}] - \overline{[X,\ Y]} - (D_X\ F)(Y) \qquad ...(2.4a)$$

Interchanging X and Y in (2.4a), we get

$$(B_Y\ F)(X) = -\ \overline{[Y,\ X]} + [Y,\ \bar{X}] - (D_Y\ F)(X). \qquad ...(2.4b)$$

Adding (2.4a) and (2.4b) and using (1.3), we get

$$(B_X\ F)(Y) + (B_Y\ F)(X) = [X,\ \bar{Y}] + [Y,\ \bar{X}].$$

If B is an M-connexion, we have (2.3).

Theorem 2.3 — If D is an *O-connexion, the condition for connexion B to be an *O-connexion is

$$[X, \overline{Y}] - \overline{[X, Y]} = \overline{[\overline{X}, \overline{Y}]} + [\overline{X}, Y]. \qquad ...(2.5)$$

PROOF : Barring Y in (2.1) and using (1.1), we get

$$(B_X F)(Y) + \overline{B_X Y} = -(D_X F)(Y) - \overline{D_X Y} + [X, \overline{Y}],$$

which on the use of (2.1) yields

$$(B_X F)(Y) + \overline{[X, Y]} = -(D_X F)(Y) + [X, \overline{Y}]. \qquad ...(2.6a)$$

Barring X and Y in (2.6a) and using (1.1), we have

$$(B_{\overline{X}} F)(\overline{Y}) + [\overline{X}, \overline{Y}] = -(D_{\overline{X}} F)(\overline{Y}) - [\overline{X}, Y]. \qquad ...(2.6b)$$

Adding (2.6a) and (2.6b) and using (1.4), we get

$$(B_X F)(Y) + (B_{\overline{X}} F)(\overline{Y}) = [X, \overline{Y}] - \overline{[X, Y]} - [\overline{X}, \overline{Y}] - [\overline{X}, Y].$$

If B is an $*O$-connexion, we have (2.5).

Theorem 2.4 — If D and B both are M-connexions, B is an $*O$-connexion.

PROOF : Barring Y in (2.3) and using (1.1), we get

$$[\overline{X}, \overline{Y}] + [X, Y] = 0. \qquad ...(2.7a)$$

Barring (2.7a), we get

$$\overline{[\overline{X}, \overline{Y}]} + \overline{[X, Y]} = 0. \qquad ...(2.7b)$$

Adding (2.3) and (2.7b), we get the required result.

Let us define a vector valued bilinear function C^* by the relation

$$C^*(X, Y) \overset{def}{=} B_X Y - B_Y X. \qquad ...(2.8)$$

On the consequence of equation (2.1), (2.8) can be written as

$$C^*(X, Y) = -S(X, Y) + [X, Y] \qquad ...(2.9)$$

which shows that C^* is skew symmetric.

It can be shown that

$$C^*(X, Y) = D_Y X - D_X Y + 2[X, Y] \qquad ...(2.10)$$

$$\overline{C^*(\overline{X}, Y)} - \overline{C^*(\overline{Y}, X)} = \overline{D_Y \overline{X}} - \overline{D_X \overline{Y}} + \overline{D_{\overline{Y}} X}$$

$$- \overline{D_{\overline{X}} Y} + 2\,\overline{[\overline{X}, Y]} - 2\overline{[\overline{Y}, X]} \qquad ...(2.11)$$

$$C^*(\bar{X}, \bar{Y}) = \overline{D_{\bar{Y}} X} - D_X Y + 2[\bar{X}, \bar{Y}] \quad ...(2.12)$$

$$C^*(X, \bar{Y}) - C^*(Y, \bar{X}) = D_{\bar{Y}} X - D_{\bar{X}} Y + \overline{D_Y X} + \overline{D_X Y} + 2[X, \bar{Y}] - 2[Y, \bar{X}]. \quad ...(2.13)$$

$$\overline{C^*(X, \bar{Y})} - \overline{C^*(Y, \bar{X})} = \overline{D_{\bar{Y}} X} - \overline{D_{\bar{X}} Y} - D_Y X + D_X Y + 2\overline{[X, \bar{Y}]} - 2\overline{[Y, \bar{X}]}. \quad ...(2.14)$$

Theorem 2.5 — If B and D are F-connexions,

$$C^*(X, \bar{Y}) = D_{\bar{Y}} X - \overline{D_X Y} + 2[X, Y]. \quad ...(2.15)$$

PROOF : Barring Y in (2.8) and using (1.1), we get

$$C^*(X, \bar{Y}) = (B_X F)(Y) + \overline{B_X Y} - B_{\bar{Y}} X,$$

which on the use of (2.1) and (2.2) yields (2.15).

Similarly we can prove the following theorems :

Theorem 2.6 — If B and D are M-connexions,

$$C^*(X, \bar{Y}) + C^*(Y, \bar{X}) = D_{\bar{X}} Y + D_{\bar{Y}} X - \overline{D_X Y} - \overline{D_Y X}. \quad ...(2.16)$$

Theorem 2.8 — If B and D are $*O$-connexions,

$$C^*(X, \bar{Y}) - C^*(\bar{X}, Y) = D_{\bar{Y}} X - D_{\bar{Y}} \bar{X} - \overline{D_X Y} - \overline{D_{\bar{X}} \bar{Y}} + 2[X, \bar{Y}] - 2[\bar{X}, Y] \quad ...(2.17)$$

3. NIJENHUIS TENSOR

A vector valued bilinear function N in an almost complex manifold, given by (Yano, 1965)

$$N(X, Y) \overset{def}{=} D_{\bar{X}} \bar{Y} - D_{\bar{Y}} \bar{X} - D_X Y + D_Y X + \overline{D_{\bar{Y}} X} - \overline{D_X \bar{Y}} - \overline{D_{\bar{X}} Y} + \overline{D_Y \bar{X}} \quad ...(3.1)$$

is called Nijenhuis tensor with respect to connexion D. Let N^* be the Nijenhuis tensor with respect to connexion B. Now we shall study the properties of N and N^* in an almost complex manifold.

Theorem 3.1 — If D is an F-connexion,

$$N(X, Y) = 0. \qquad ...(3.2)$$

PROOF : Using (1.1) and (1.2) in (3.1), we get (3.2).

Theorem 3.2 — If D is an F-connexion,

$$N^*(X, Y) = 2([\bar{X}, \bar{Y}] - [X, Y] - \overline{[X, \bar{Y}]} - \overline{[\bar{X}, Y]}). \qquad ...(3.3)$$

PROOF : Using (2.1), (1.1) and (1.2) and the definition of $N^*(X, Y)$, we get (3.3).

Theorem 3.3 — If D is an F-connexion, then N^* is hybrid (Yano 1965).

PROOF : The proof is obvious.

Theorem 3.4 — We have

$$N^*(X, Y) = C^*(\bar{X}, \bar{Y}) - C^*(X, Y) - \overline{C^*(X, \bar{Y})} - \overline{C^*(\bar{X}, Y)}. \qquad ...(3.4)$$

PROOF : Using (2.8) and the definition of $N^*(X, Y)$, we get (3.4).

4. TENSOR M

Let us consider a vector valued bilinear function M given by (Mishra 1969, Sinha 1970)

$$M(X, Y) \overset{def}{=} D_{\bar{X}} \bar{Y} + D_X Y - \overline{D_{\bar{X}} Y} - \overline{D_X \bar{Y}}. \qquad ...(4.1)$$

for the connexion D. Let M^* be the vector valued bilinear function for the connexion B. Now we shall investigate some of their properties.

Theorem 4.1 — If D is an F-connexion,

$$M(X, Y) = 2\, D_X\, Y. \qquad ...(4.2)$$

PROOF : Using (1.1) and (1.2) in (4.1), we get (4.2)

Theorem 4.2 — If B and D are M-connexions,

$$M^*(X, Y) + M(X, Y) = -\,2\overline{[\bar{X}, Y]}. \qquad ...(4.3)$$

PROOF : Using (2.1) and the definition of $M^*(X, Y)$, we get

$$M^*(X, Y) = -\,D_{\bar{X}}\, \bar{Y} + [\bar{X}, \bar{Y}] - D_X\, Y + [X, Y] + \overline{D_{\bar{X}}\, Y} - [\bar{X}, Y]$$

$$+\, \overline{D_X\, \bar{Y}} - [X, \bar{Y}],$$

which on the use of (4.1) and (2.3) yields (4.3).

Similarly we can obtain the following results :

Theorem 4.3 — If B and D are F-connexions,

$$M^*(X, Y) = -2 D_X Y + 2[X, Y]. \qquad ...(4.4)$$

Theorem 4.4 — If B and D are $*O$-connexions,

$$M^*(X, Y) + M(X, Y) + 2[X, \bar{Y}] = 0. \qquad ...(4.5)$$

Corollary 4.1 — If B is an F-connexion,

$$M^*(X, Y) + M^*(Y, X) + 2(D_X Y + D_Y X) = 0. \qquad ...(4.6)$$

Theorem 4.6 — If D is an F-connexion,

$$M^*(X, Y) - M^*(Y, X) = N^*(X, Y) - 2S(X, Y) + 2[X, Y]. \qquad ...(4.7)$$

PROOF : Using (2.1), (1.1) and (1.2) and the definition of $M^*(X, Y)$ we get

$$M^*(X, Y) = -2D_X Y + [X, Y] + [\bar{X}, \bar{Y}] - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]}. \qquad ...(4.8a)$$

Interchanging X and Y in (4.8a), we get

$$M^*(Y, X) = -2D_Y X + [Y, X] + [\bar{Y}, \bar{X}] - [\bar{Y}, X] - \overline{[Y, \bar{X}]}. \qquad ...(4.8b)$$

Subtracting (4.8b) from (4.8a), we get (4.7) on the use of (3.3).

Corollary 4.2 — We have

$$M^*(X, Y) - M^*(Y, X) = N^*(X, Y) + 2C^*(X, Y). \qquad ...(4.9)$$

PROOF : Using (2.9) in (4.7), we get (4.9).

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