ON THE CONVERGENCE OF SERIES OF LEGENDRE POLYNOMIALS

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The authors have obtained the necessary and sufficient Tauberian conditions for the criteria for the convergence of Legendre series. However, they also obtain two results, which are used in the above criteria, concerning Riesz summability of Legendre series of order unity and types $\exp(n^{\alpha})$ $(0<\alpha<1)$ and $\exp\{(\log n)^{\Delta}\}$ $(\Delta > 1)$. All the results obtained in the paper are new.

1. DEFINITIONS AND NOTATIONS

Let $\sum_{n=0}^{\infty} b_n$ be a given infinite series with the sequence $\{S_n\}$ of partial sums, where

$$S_n = b_0 + b_1 + b_2 + \ldots + b_n.$$

Throughout the paper we suppose that

$$0 < \lambda_n = \mu_0 + \mu_1 + \mu_2 + \ldots\ldots + \mu_n \to \infty \text{ ; as } n \to \infty. \qquad \ldots(1.1)$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{\lambda_n} \sum_{m=0}^{n} \mu_m S_m. \qquad \ldots(1.2)$$

defines the Riesz means of the sequence $\{S_n\}$ (or the series $\sum_{n=0}^{\infty} b_n$) of the type λ_{n-1} and order unity*. If $t_n \to S$; as $n \to \infty$, the sequence $\{S_n\}$ is said to be summable by Riesz means of order unity and type λ_{n-1} or symbolically we write

*It is some times called $(\bar{N}, \mu_n)$ mean, or (R, μ_n) mean, or Riesz's discrete means of 'type' λ_{n-1} and 'order' unity and is infact, equivalent to the usually known $(R, \lambda_{n-1}, 1)$ mean. See Hardy (1949, p. 57). Reference may also be made to Chandra (1970).

$\{S_n\} \in (R, \lambda_{n-1}, 1)$ to the sum S.

Let function $f \in L\,[-1, 1]$. Then the Legendre series associated with the function f, at an internal point $x \in [-1, 1]$, will be

$$\sum_{n=0}^{\infty} a_n P_n(x) \qquad \ldots(1.3)$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} f(y) P_n(y)\, dy \qquad \ldots(1.4)$$

and $P_n(y)$ denotes the Legendre polynomials defined by the relation

$$(1 - 2\rho y + \rho^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \rho^n P_n(y) \quad (\,|\,\rho\,| < 1). \qquad \ldots(1.5)$$

We use the following notations :

$$M(u) = \{f(x \pm u) - f(x)\} \qquad \ldots(1.6)$$

$$f(\theta, \phi) = \{f(\cos\theta) - f(\cos\phi)\} \qquad \ldots(1.7)$$

$$\psi(t) = [f\{\cos(\theta \pm t)\} - f(\cos\theta)] \qquad \ldots(1.8)$$

$$\Psi(t) = \int_{0}^{t} |\,\psi(u)\,|\; du. \qquad \ldots(1.9)$$

2. Introduction

The object of this paper is to obtain two criteria for the convergence of Legendre series at an internal point $x \in [-1, 1]$.

The technique of the proof of the above criteria will be to obtain the theorems (Theorems 3 and 4) on the Riesz summability of Legendre series at an internal point $x \in [-1, 1]$, and to deduce Theorems 1 and 2 by means of a Tauberian theorem (Lemma 5) obtained by the present authors.

Precisely, we prove the following :

Theorem 1 — Let

$$f(u)(1 - u^2)^{-\frac{1}{4}} \in L\,[-1, 1], \qquad \ldots(2.1a)$$

$$\int_0^t |M(u)|\, du = o\left(\frac{t}{\log \frac{1}{t}}\right);\ \text{as } t \to 0. \qquad \text{...(2.1}b\text{)}$$

Then the necessary and sufficient condition for the convergence of the series (1.3), at any internal point $u = x \in [-1, 1]$, to $f(x)$, is that

$$\exp\{-(n+1)^{\alpha}\} \sum_{m=1}^{n} a_m P_m(x) \exp(m^{\alpha}) = o(1); \qquad \text{...(2.2)}$$

as $n \to \infty$, where $0 < \alpha < 1$.

Theorem 2—Let (2.1*a*) hold and let

$$\int_0^t |M(u)|\, du = o\left(\frac{t}{\log \log \frac{1}{t}}\right),\ as\quad t \to 0. \qquad \text{...(2.3)}$$

Then the necessary and sufficient condition for the convergence of the series (1.3), at any internal point $u = x \in [-1, 1]$, to $f(x)$, is that

$$\exp\left\{-(\log(n+1))^{\Delta}\right\} \sum_{m=1}^{n} \exp\left\{(\log m)^{\Delta}\right\} a_m P_m(x) = o(1) \qquad \text{...(2.4)}$$

as $n \to \infty$, where $\triangle > 1$.

Theorem 3—Let (2.1) hold. Then the series (1.3) $\in (R, e^{n^{\alpha}}, 1)$ $(0 < \alpha < 1)$, at any internal point $u = x \in [-1, 1]$, to $f(x)$.

Theorem 4—Let (2.1*a*) and (2.3) hold.

Then the series (1.3) $\in (R, \exp\{(\log n)^{\Delta}\}, 1)$ $(\triangle > 1)$, at any internal point

$u = x \in [-1, 1]$ to $f(x)$.

3. Lemmas

We require the following lemmas for the proof of the theorems.

Lemma 1—Let $0 < \theta < \pi$. Then (2.1) and (2.3) respectively imply that

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left(\frac{t}{\log \frac{1}{t}}\right),\ \text{as } t \to 0 \qquad \text{...(3.1)}$$

and

$$\Psi(t) = \int_0^t |\psi(u)|\,du = \delta\left(\frac{t}{\log\log\frac{1}{t}}\right), \text{ as } t \to 0. \qquad \ldots(3.2)$$

The proof of the lemma runs on the lines of Foa (1943). Also see Gupta (1957).

Lemma 2—Let $0 < \theta < \pi$ and $0 < \phi < \pi$ and let η be a fixed positive constant less than $\pi/2$ such that $0 \leqslant \theta - \eta < \theta + \eta \leqslant \pi$.

Then, as $n \to \infty$,

$$D_n(\theta) = \int_{\theta-\eta}^{\theta+\eta} f(\theta, \phi)\, \frac{\sin (n+1)(\theta-\phi)}{\sin \frac{(\theta-\phi)}{2}}\, \sin^{\frac{1}{2}} \phi \, d\phi = I + o(1)$$

where

$$I = \int_{-\eta}^{\eta} \psi(t)\, \frac{\sin nt}{\sin \frac{t}{2}}\, \sin^{\frac{1}{2}}(\theta - t)\, dt.$$

PROOF : On putting $\theta - \phi = t$, we have

$$D_n(\theta) = \int_{-\eta}^{\eta} \psi(t)\, \frac{\sin (n+1)\, t}{\sin \frac{t}{2}}\, \sin^{\frac{1}{2}}(\theta - t)\, dt$$

$$= \int_{-\eta}^{\eta} \psi(t)\, \frac{\sin nt}{\sin \frac{t}{2}}\, \sin^{\frac{1}{2}}(\theta - t)\, dt$$

$$+ 2\int_{-\eta}^{\eta} \psi(t) \cos (n + \tfrac{1}{2})\, t \sin^{\frac{1}{2}}(\theta - t)\, dt$$

$$= \int_{-\eta}^{\eta} \psi(t)\, \frac{\sin nt}{\sin \frac{t}{2}}\, \sin^{\frac{1}{2}}(\theta - t)\, dt + o(1)$$

as $n \to \infty$, by Riemann-Lebesgue Theorem.

Lemma 3 — Let (2.1*a*) hold and let

$$R_n(x) = S_n(x) - f(x).$$

Then, by transformation

$$x = \cos\theta \; (0 < \theta < \pi) \text{ and } y = \cos\phi \; (0 < \phi < \pi)$$

we have

$$R_n(x) = \frac{1}{2\pi} I + o(1), \text{ as } n \to \infty$$

where I be the integral defined in Lemma 2.

PROOF : By using the transformation $x = \cos\theta$ and $y = \cos\phi$, we have (see Foa 1943 and Sansone 1959, p. 225)

$$R_n(x) = \frac{1}{2\pi} D_n(\theta) + o(1), \text{ as } n \to \infty,$$

where $D_n(\theta)$ is as defined in Lemma 2. Further, by using Lemma 2, we follow the proof of Lemma 3.

Lemma 4 — Let $0 < \eta' < \eta < \pi$, such that $0 < \theta - \eta \leqslant \pi$.
Then ; as $n \to \infty$,

$$\int_0^{\eta} \psi(t) \frac{\sin nt}{\sin \frac{t}{2}} \sin^{\frac{1}{2}}(\theta - t)\, dt$$

$$= 2\int_0^{\eta'} \psi(t) \frac{\sin nt}{t} \sin^{\frac{1}{2}}(\theta - t)\, dt + o(1).$$

PROOF : we have

$$\int_0^{\eta} \psi(t) \frac{\sin nt}{\sin\frac{t}{2}} \sin^{\frac{1}{2}}(\theta - t)\, dt = \int_0^{\eta} \psi(t) \sin nt \left[\frac{1}{\sin\frac{t}{2}} - \frac{1}{\frac{t}{2}}\right]$$

$$\times \sin^{\frac{1}{2}}(\theta - t)\, dt + 2\int_0^{\eta} \psi(t) \frac{\sin nt}{t} \sin^{\frac{1}{2}}(\theta - t)\, dt$$

$$= L + M, \text{ say.}$$

Now,

$$L = o(1), \text{ as } n \to \infty,$$

by Riemann-Lebesgue theorem.

Now we write, for $0 < \eta' < \eta$,

$$M = 2\left[\int_0^{\eta'} + \int_{\eta'}^{n}\right]\left(\psi(t)\frac{\sin nt}{t}\sin^{\frac{1}{2}}(\theta - t)\,dt\right)$$

$$= 2\left[M_1 + M_2\right], \text{ say.}$$

Now,

$$M_2 = \int_{\eta'}^{\eta}\frac{\psi(t)}{t}\sin nt \sin^{\frac{1}{2}}(\theta - t)\,dt$$

$$= o(1), \text{ as } n \to \infty,$$

by Riemann-Lebesgue theorem. Hence the lemma follows.

Lemma 5 — Let $\sum_{n=0}^{\infty} b_n \in (R, \lambda_{n-1}, 1)$ to S.

Then the necessary and sufficient condition, for the convergence of the series $\sum_{n=0}^{\infty} b_n$ to the sum S, is that

$$\left\{\frac{1}{\lambda_n}\sum_{m=1}^{n}\lambda_{m-1}\,b_m\right\} = o(1), \text{ as } n \to \infty. \qquad \ldots(3.3)$$

PROOF: The $(R, \lambda_{n-1}, 1)$-mean of $\sum_{n=0}^{\infty} b_n$ is defined by

$$t_n = \frac{1}{\lambda_n}\sum_{m=0}^{n}\mu_m\,S_m$$

where $\mu_n = \lambda_n - \lambda_{n-1}$ and $S_n = b_0 + b_1 + \ldots\ldots\ldots\ldots + b_n$.

By applying Abel's transformation to the right hand side, we have

$$t_n = \frac{1}{\lambda_n}\left\{\sum_{m=0}^{n-1}\left(S_m - S_{m+1} \right)\sum_{K=0}^{m} \mu_k + S_n \sum_{K=0}^{n} \mu_k \right\}.$$

$$= S_n - \frac{1}{\lambda_n}\sum_{m=1}^{n} \lambda_{m-1}\, b_m.$$

Since $\sum_{n=0}^{\infty} b_n \in (R, \lambda_{n-1}, 1)$, $t_n \to S$; as $n \to \infty$, the proof of the lemma follows from the above equation.

4. Proof of the Theorems

We shall first obtain the proofs of Theorems 3 and 4. And then we shall deduce, from Lemma 5, Theorems 1 and 2 by using, respectively, Theorems 3 and 4.

Proof of Theorem 3 — We have

$$S_n(x) = \sum_{m=0}^{n} a_m P_m(x)$$

$$= \sum_{m=0}^{n}\left(m + \frac{1}{2}\right) P_m(x) \int_{-1}^{1} f(y)\, P_m(y)\, dy$$

$$= \frac{1}{2}(n+1) \int_{-1}^{1} \frac{P_n(x)\, P_{n+1}(y) - P_n(y)\, P_{n+1}(x)}{(y-x)}\, f(y)\, dy.$$

Now the series

$$\sum_{n=0}^{\infty} a_n P_n(x) \in (R, e^{n^{\alpha}}, 1)\ (0 < \alpha < 1) \text{ to } f(x)$$

if the sequence $\{R_n(x)\} \in (R, e^{n^{\alpha}}, 1)$ to zero, where

$$R_m(x) = S_m(x) - f(x).$$

Now, by Lemmas 2 and 3, we have

$$R_m(x) = \int_{-\eta}^{\eta} \psi(t) \frac{\sin mt}{\sin \frac{t}{2}} \sin^{\frac{1}{2}}(\theta - t)\, dt + o(1), \text{ as } m \to \infty, \qquad \ldots(4.1)$$

where $\eta < \pi/2$ be the fixed positive constant such that $0 \leqslant \theta - \eta < \theta + \eta \leqslant \pi$.

Now, we write

$$I = \int_{-\eta}^{\eta} \psi(t) \frac{\sin mt}{\sin \frac{t}{2}} \sin^{\frac{1}{2}} (\theta - t)\, dt.$$

$$= \left[\int_0^{\eta} + \int_{-\eta}^{0} \right] \left(\psi(t) \frac{\sin mt}{\sin \frac{t}{2}} \sin^{\frac{1}{2}} (\theta - t)\, dt \right)$$

$$= I_1 + I_2, \text{ say.}$$

Then, from (3.1), given $\epsilon > 0$; there exists η', depending upon ϵ, such that

$$\Psi(t) < \epsilon \frac{t}{\log (1/t)}, \text{ for } 0 < t < \eta'. \qquad \ldots(4.2)$$

On choosing $\eta' < \eta$ and applying Lemma 4 we have, as $m \to \infty$,

$$I = 2 \int_0^{\eta'} \psi(t) \frac{\sin mt}{t} \sin^{\frac{1}{2}} (\theta - t)\, dt + o(1).$$

And, on writing

$$J = \int_0^{\eta'} \psi(t) \frac{\sin mt}{t} \sin^{\frac{1}{2}} (\theta - t)\, dt$$

$$= \left[\int_0^{1/n} + \int_{1/n}^{1/n^{\beta}} + \int_{1/n^{\beta}}^{n'} \right]$$

$$= J_1 + J_2 + J_3, \text{ say,}$$

where $0 < \beta < \frac{1}{2}(1-\alpha)$ $(0 < \alpha < 1)$ and $n \geqslant m$ and by using (4.2) we have

$$| J_1 | \leqslant m \int_0^{1/n} | \psi(t) | \, dt$$

$$< \epsilon m \left(\frac{1}{n \log n} \right)$$

$$= o(1) \; ; \text{ as } n \to \infty$$

and

$$| J_2 | \leqslant \int_{1/n}^{1/n^\beta} \frac{| \psi \, (t) |}{t} \, dt$$

$$= n^\beta \, \Psi \, (n^{-\beta}) - n \, \Psi \, (n^{-1}) + \int_{n^{-1}}^{n^{-\beta}} \frac{\Psi \, (t)}{t^2} \, dt$$

$$< \frac{\epsilon}{\log n} \left(1 + \frac{1}{\beta} \right) + \epsilon \int_{n^{-1}}^{n^{-\beta}} \frac{1}{t \log (1/t)} \, dt, \text{ by (4.2)}$$

$$= \epsilon \left[\frac{1 + (1/\beta)}{\log n} + \log \, (1/\beta) \right].$$

Therefore

$$\lim_{n \to \infty} \; \text{Sup} \; | J |$$

may be made arbitrarily small by choosing ϵ sufficiently small. Hence (R, e^{n^α}, 1)-transform of J_2 may be made arbitrarily small by choosing ϵ sufficiently small.

Therefore $(R, e^{n^\alpha}, 1)$-transform of I, will tend to zero, if we prove that

$$\lim_{n \to \infty} \left[e^{-(n+1)^\alpha} \int_{m-\beta}^{n} \frac{\psi \, (t)}{t} \sin^{\frac{1}{2}} (\theta - t) \right.$$

$$\left. \times \left(\sum_{m=1}^{n} \left(e^{(m+1)^\alpha} \times - e^{m^\alpha} \right) \sin mt \right) dt \right] = 0.$$

Now, by using Abel's lemma, we have

$$K\,(n, \, t) = \sum_{m=1}^{n} \left(e^{(m+1)^\alpha} - e^{m^\alpha} \right) \sin mt$$

$$= O \left\{ \left(e^{(n+1)^\alpha} - e^{n^\alpha} \right) t^{-1} \right\}$$

$$= O\left\{\frac{n^{\alpha-1}\, e^{(n+1)^{\alpha}}}{t}\right\},$$

uniformly in $0 < t < \pi$. Thus

$$e^{-(n+1)^{\alpha'}} \int_{n^{-\beta}}^{\eta'} \frac{\psi(t)}{t} \sin^{\frac{1}{2}}(\theta - t)\, K(n, t)\, dt.$$

$$= O\left\{ n^{\alpha-1} \int_{n^{-\beta}}^{\eta'} \frac{|\psi(t)|}{t^2}\, dt \right\}$$

$$= O\left\{ n^{\alpha+2\beta-1} \int_{n^{-\beta}}^{\eta'} |\psi(t)|\, dt \right\},$$

which tends to zero as $n \to \infty$. This proves that $(R, e^{n^{\alpha}}, 1)$-transform of I_1 tends to zero as $n \to \infty$. Similarly we can prove that $(R, e^{n^{\alpha}}, 1)$-transform of I_2 tends to zero as $n \to \infty$.

Thus combining these results with the right hand side of (4.1), we follow that $(R, e^{n\alpha}, 1)$-transform of $\{R_n(x)\}$ tends to zero as $n \to \infty$. This proves the theorem 3.

Proof of Theorem 4—Proceeding as in the proof of Theorem 3 and breaking up the integral, corresponding to J, into the integrals, J_1, J_2 and J_3 over $\left(0, \frac{1}{n}\right)$, $\left(\frac{1}{n}, \frac{(\log n)^{\Delta_1}}{n}\right)$ and $\left(\frac{(\log n)^{\Delta_1}}{n}, \eta'\right)$ respectively, where $\Delta_1 > \Delta$ and η' is as defined below, by using (3.2);

Given $\epsilon > 0$; there exists η', depending upon ϵ, such that

$$\Psi(t) < \epsilon \frac{t}{\log \log (1/t)}, \text{ for } 0 < t < \eta', \qquad \ldots(4.3)$$

we have

$$J_1 = o(1), \text{ as } n \to \infty.$$

Now $\lim\limits_{n\to\infty} \operatorname{Sup} | J_2 |$

is arbitrarily small on choosing ϵ sufficiently small. And $(R, \exp\{(\log n)^{\Delta}\}, 1)$-transform of J_3 is

$$P_n = \frac{1}{\exp\{(\log (n+1)^{\Delta}\}} \sum_{m=1}^{n} \left(\exp\{(\log(m+1))^{\Delta}\} - \{(\log m)^{\Delta}\} \right) J_3.$$

Now, to prove $P_n = o(1)$, as $n \to \infty$, it is sufficient to prove that,

$$Q_n = \exp\left\{-(\log(n+1))^{\Delta}\right\} \int_{\frac{(\log n)^{\Delta_1}}{n}}^{\eta'} \frac{\psi(t)}{t} \sin^{\frac{1}{2}}(\theta - t)\, E(n, t)\, dt.$$

$$= O(1), \text{ as n} \to \infty,$$

where

$$E(n, t) = \sum_{m=1}^{n} \Big[\exp\{(\log(m+1))^{\Delta}\} - \exp\{(\log m)^{\Delta}\} \Big] \sin mt.$$

Since $\Delta > 1$, $[\exp\{(\log(m+1))^{\Delta}\} - \exp\{(\log m)^{\Delta}\}]$ is monotonic increasing with m for $\geqslant m_0$ (some finite number). Therefore, by using Abel's lemma, we have

$$E(n, t) = O\left\{\frac{\exp\{(\log(n+1))^{\Delta}\} - \exp\{(\log n)^{\Delta}\}}{t}\right\}$$

$$= O\left\{\frac{(\log n)^{\Delta-1} \exp\{(\log(n+1)^{\Delta}\}}{nt}\right\}. \qquad \ldots(4.4)$$

Thus, by (4.2.2), we have

$$Q_n = O\left\{\frac{(\log n)^{\Delta-1}}{n} \int_{\frac{(\log n)^{\Delta_1}}{n}}^{\eta'} \frac{|\psi(t)|}{t^2}\, dt\right\}$$

$$= O\left\{\frac{(\log n)^{\Delta-1}}{(\log n)^{\Delta_1}} \int_{\frac{(\log n)^{\Delta_1}}{n}}^{\eta'} \frac{|\psi(t)|}{t}\, dt\right\}.$$

Now, integrating by parts and using (4.3), we have

$$\int_{\frac{(\log n)^{\Delta_1}}{n}}^{\eta'} \frac{|\psi(t)|}{t}\, dt$$

$$= \Big[t^{-1}\Psi(t)\Big]_{\frac{(\log n)^{\Delta_1}}{n}}^{\eta'} + \int_{\frac{(\log n)^{\Delta_1}}{n}}^{\eta'} \frac{\Psi(t)}{t^2}\, dt$$

$$= o(\log n).$$

Therefore

$$Q_n = O\left\{\frac{(\log n)^{\triangle}}{(\log n)^{\triangle_1}}\right\},$$

which tends to zero; as $n \to \infty$, whenever $\triangle_1 > \triangle$. This proves that $(R, \exp\{(\log n)^{\triangle}\}, 1)$-transform of J and therefore the integral corresponding to I_1 of Theorem 3, tends to zero as $n \to \infty$. Similarly the integral corresponding to I_2 of Theorem 3, can be disposed of.

This terminates the proof of Theorem 4.

Proof of Theorems 1 *and* 2—By Theorem 3,

$$\sum_{n=0}^{\infty} a_n P_n(x) \in (R, e^{n^{\alpha}}, 1)\ (0 < \alpha < 1).$$

Now by using Lemma 5 for $\lambda_n = e^{n^{\alpha}}$ $(0 < \alpha < 1)$ and $b_n = a_n P_n(x)$, we follow the proof of Theorem 1.

Similarly, since

$$\sum_{n=0}^{\infty} a_n P_n(x) \in (R, \exp\{(\log n)^{\triangle}\}, 1)\ (\triangle > 1)$$

by Theorem 4, we follow the proof of Theorem 2 by using Lemma 5 again for $\lambda_n = \exp\{(\log n)^{\triangle}\}$, $\triangle > 1$ and $b_n = a_n P_n(x)$.

This completes the proof of Theorems 1 and 2.

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