ON NON-SINGULAR RINGS

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The object of this paper is to characterize non-singular rings including semi-simple rings and to study their properties. The following results have been obtaind :

(1) A ring R with unit is non-singular if no maximal left ideal is essential. (2) If R is a ring with unit then any proper minimal left ideal of R is non-singular if and only if it is generated by an idempotent. (3) For a polysimple ring R the following conditions are equivalent.

(a) $Z(R) = 0$ and every projective left ideal is direct summand.

(b) R is semi-simple.

(c) Every minimal left ideal is a direct summand and R satisfies FFC.

(d) Every simple R-module is non-singular.

(e) Every maximal ideal is a direct summand.

(4) Over a semi-prime quasi-dedekind Goldie ring R every singular module is injective. (5) Let M be a singular R-module over a semi-prime Goldie ring R and N proper module extenion of M. Then N is singular if and only if for every $x \varepsilon N \setminus M$ there exists a regular element, $\alpha \varepsilon R$ such that $\alpha x \varepsilon M$.

INTRODUCTION

In this paper characterizations have been given for (1) non-singular simple modules, (2) non-singular minimal left ideas of arbitrary rings with unit and (3) semi-simple rings with d.c.c. In addition it has been shown that every singular module over semi-prime quasi-dedekind Goldie ring is injective. A necessary and sufficient condition for extension of a singular module over a semi-prime Goldie ring to be singular has also been provided.

DEFINITIONS AND NOTATIONS

All rings considered in this paper are supposed to be with unit and every module is supposed to be unitary.

Definition 1—A module M is said to be polysimple if every non-zero submodule of it contains a simple submodule.

Definition 2—A ring R is said to be a quasi-dedekind ring if each of its regular ideals is a product of prime ideals.

Definition 3—The singular submodule $Z(M)$ of an R-module M is the collection of all the elements of M whose order ideals are essential in R. The singular ideal $Z(R)$ of R is the singular R-submodule of R. A module M is said to be singular or non-singular according as $Z(M) = M$ or $Z(M)=\theta$.

Definition 4—A commutative non-singular ring R, over which for every R-module M, $Z(M)$ is a direct summand of M, is said to be a ring with S.P. We shall express the fact that N is essential extension of M by $N \triangle M$, $J(R)$ will denote the Jacobson radical of R. $O(x)$ will denote the order ideal of x. The semi-simple rings considered in this paper will be supposed to satisfy d.c.c. for left ideals.

Lemma 1—If M is a direct sum of a family of submodules $(M_i)_{i\varepsilon I}$ then $Z(M) = \sum_{i \varepsilon I} Z(M_i)$. Hence M is non-singular if and only if each M_i is non-singular.

PROOF : If for any $x \in M$, $x=\Sigma x_i$, $x_i \in M_i$, $\alpha x=0$ then $\alpha x_i=0$ for each $i \in I$. This implies that $O(x) \subseteq \bigcap_{i \varepsilon I} O(x_i)$. In particular if $x \in Z(M)$ then $R \triangle O(x)$ implies $R \triangle O(x_i)$ for each $i \in I$ and so $x_i \in Z(M_i)$. Hence $Z(M)=\Sigma Z(M_i)$. It follows that $Z(M)$ is zero if and only if $Z(M_i)=0$ for each $i \in I$.

Remark 1 : If R is a ring with unit 1 such that no maximal left ideal is essential, then R is non-singular.

PROOF : Suppose $Z(R) \neq 0$. Then there exists a non-zero element $\alpha \in R$ with $R \triangle O(\alpha)$.

Now, $1.\alpha = \alpha \neq 0$ implies that $1 \notin O(\alpha)$, so R is a proper essential extension of $O(\alpha)$. Hence there exists a maximal left ideal $M \supseteq O(\alpha)$. Clearly $R \triangle M$, but this contradicts the fact that no maximal ideal is an essential ideal. Hence R is non-singular.

Theorem 1—Let R be a ring with unit 1, M any maximal left ideal of R. Then the following conditions are equivalent :

(1) M is a direct summand of R.

(2) $R \not\triangle M$.

(3) $R \mid M$ is non-singular.

PROOF : (1) *implies* (2): If M is a direct summand of R, then for some left ideal L of R, $M \cap L=0$. Hence $R \not\triangle M$.

(2) *implies* (3) : Suppose $R \not\triangle M$. Then $M \cap L = 0$ for some non-zero left ideal L of R, so that

$$R = M + L \text{ (direct)}$$

L being a direct summand of the projective module R, $R \mid M$ simple and projective. Hence non-singular.

(3) *implies* (1) : If $R \mid M$ is non-singular, then it is projective (Fulberth and Teply 1972). Hence M is a direct summand of R.

Corollary—If a maximal left ideal is essential ideal of R, then $R \mid M$ is singular.

PROOF : R essential extension of M implies $Z(R \mid M) \neq 0$, hence $Z(R \mid M) = R \mid M$ since $R \mid M$ is simple.

Lemma 2—Let R be a ring and L any proper minimal left ideal of R. Then L is non-singular if and only if L is generated by an idempotent.

PROOF : Let L be non-singular, then since $L \cong R \mid K$ for some maximal left ideal K of R, L is a direct summand of R. Hence

$$R = L + K \quad \text{(direct)}.$$

Let $1 = e + e'$ where $e \in L$ and $e' \in K$, then

$$e = e^2 + ee'$$

so

$$e - e^2 = ee' \in L \cap K.$$

This implies that $e = e^2$, therefore $L = Re$.

Conversely, if $L = Re$ where $e = e^2$, then

$$R = Re + R(1 - e) \quad \text{(direct)}$$

R being projective, Re is projective, so $Z(Re) = 0$.

Lemma 3—Every semi-prime commutative ring R with unit 1 is non-singular.

PROOF : Suppose $Z(R) \neq 0$, then there exists $x\ (\neq 0) \in Z(R)$ such that $R \triangle O(x)$. Now, for any $\alpha \in O(x)$, $\beta x \in Rx$, we have

$$\alpha\beta x = \beta\alpha x = 0.$$

This implies that $O(x).\ Rx = 0$ (Lambek 1966). Hence by semi-primeness of R,

$$O(x) \cap (Rx) = 0.$$

Therefore $Rx = 0$ which is a contradiction. Hence $Z(R) = 0$.

Theorem 2—For a polysimple ring R with unit, the following conditions are equivalent :

(1) $Z(R) = 0$ and every projective left ideal is a direct summand.

(2) R is semi-simple.

(3) R satisfies FFC and every minimal left ideal is a direct summand.

(4) Every simple R-module is non-singular.

(5) Every maximal left ideal is a direct summand.

PROOF : (1) *implies* (2) : Since $Z(R) = 0$, every minimal left ideal of R is non-singular and hence projective. It follows that Soc(R) is projective. By (1)

$$R = \mathrm{Soc(R)} + A \quad \text{(direct)}$$

for some left ideal A of R. But $R \triangle \mathrm{Soc}(R)$ and

$$\mathrm{Soc}(R) \cap A = 0$$

hence $A = 0$ implies $R = \mathrm{Soc}(R)$ is semi-simple.

(2) *implies* (3) : R semi-simple $\Rightarrow$ R satisfies d.c.c. and hence FFC (Tiwary and Pandey 1971). Also by Lemma 1, R is non-singular since R is a direct sum of minimal left ideals each of which is non-singular (Cheathem 1971).

(3) *implies* (4) : Consider any simple module $S = Rx \cong R \mid O(x)$. Suppose $Z(R \mid O(x)) \neq 0$, then since $O(x)$ is a maximal left ideal of R, by Theorem, 1 $R \triangle O(x)$. If L is any minimal left ideal of R, then $L \cap O(x) \neq 0 \Rightarrow L \subseteq O(x) \Rightarrow \mathrm{Soc}(R) \subseteq O(x)$. By FFC on R, Soc(R) is a direct sum of finitely many minimal left ideals of R, each of which is non-singular and so is generated by an idempotent (Lemma 2). Hence $\mathrm{Soc}(R) = Re$ for some idempotent e in R, is a direct summand of R. This gives (Gordon 1969)

$$R = \mathrm{Soc}(R) = O(x)$$

$$\Rightarrow S = 0, \text{ a contradiction.}$$

Hence S is non-singular.

(4) *implies* (5) : $M \subset R$ maximal left ideal implies that $R \mid M$ is simple and so $Z(R \mid M) = 0$. Therefore $R \mid M$ is projective whence M is a direct summand of R.

(5) *implies* (1) : Suppose $Z(R) \neq 0$. Then there exists a non-zero $\alpha \in R$ with $R \triangle O(\alpha)$, $O(\alpha) \subset R$ since $1 \notin O(\alpha)$. Therefore $O(\alpha)$ is contained in a maximal left ideal. M, whence $R \triangle M \Rightarrow M$ is not a direct summand of R by theorem 1, contradicting (5). Hence R is non-singular. Clearly $R = \mathrm{Soc}(R)$ for otherwise Soc(R) would be contained in a maximal left ideal which would be essential in R and hence not a direct summand of R contrary to (5). Hence every left ideal of R in particular every projective one is a direct summand of R.

Corollary 1—Every module over a polysimple ring R is projective if and only if it is non-singular.

Corollary 2—A ring R is semi-simple if and only if R is regular polysimple and satisfies FFC.

Corollary 3—A commutative ring R is semi-simple if and only if R is a polysimple ring and satisfies FFC.

Theorem 3—A commutative polysimple ring with unit 1 having a socle of finite length is semi-simple.

PROOF : Soc(R) being semi-simple, every minimal left ideal is generated by an idempotent. This implies that $\text{Soc}(R) = Re_1 + \ldots + Re_n$ (dir) where e_i are idempotents different from zero. By commutativity of R, $e = e_1 + \ldots + e_n$ is an idempotent and $R = \text{Soc}(R) + R(1-e)$. If $a = \alpha e_1 + \ldots + \alpha_n e_n = \beta - \beta e \in \text{Soc}(R) \cap R(1-e)$, then $ae = \beta e - \beta e = 0$. This implies that $\text{Soc}(R) \cap R(1-e) = 0$ and so $R = \text{Soc}(R) + R(1-e)$ (dir.). Since $R \triangle \text{Soc}(R)$ one has $R(1-e) = 0$. Hence $R = \text{Soc}(R)$ is semi-simple.

Theorem 4—Let R be a polysimple ring and let E be an extension ring of R such that E is self injective and $E \triangle R$ as an R-module. If either E or R is semi-prime then E is semi-primitive.

PROOF : Suppose E is semi prime. Then E being an essential extension of a polysimple module R is poly simple as an R module, hence $Z_E(E) = 0$ (Tiwary and Pandey 1975). By Theorem 2, E is semi-simple and so E is semi-primitive. Again if R is semi-prime and polysimple, then $Z_R(R) = 0$ and so by Theorem 2, R is semi-simple. $E \triangle R$ implies that $Z_R(E) = 0$ and E is injective as an R module. Therefore by (Faith and Utumi 1964) $Z_E(E) = 0$. Since E is self injective, therefore, $Z_E(E) = J(E) = 0$ so E is semi-primitive.

Theorem 5—Over a semi-prime quasi-dedekind Goldie ring R every singular module is injective.

PROOF : Let M be a singular module over a quasi dedekind semi prime Goldie ring R. Suppose M is not injective. Then there exists an injective hull N of M with $N \supset M$. Now $N \triangle M$ implies that there exists an element $x \in N \setminus M$ with $0 \neq \alpha x \in M$ for some $\alpha \in R$. Hence $O(\alpha x)$ is an essential ideal of R. Let $I = O(\alpha x)$. Since I is essential ideal of R, I contains a regular element [Goldie 1960]. R quasi-dedekind ring implies that I is an invertible ideal of R [Singh and Wasan 1970], so there exists a R-submodule B of the total quotient ring Q of R such that $BI(\alpha x) = R\alpha x = 0$. Since R contains a unit element 1 therefore $\alpha x = 0$ which is a contradiction, hence M is an injective R-module.

Corollary 1—Over a dedekind ring R every singular R-module is injective.

Corollary 2—Over a multiplicative semi-prime Goldie ring R every singular R-module is injective.

Corollary 3—Over a hereditary semi-prime Goldie ring R every singular R-module is injective.

Recall that a commutative ring R with unit element 1 has S.P. if and only if every singular R-module is injective.

Remark 2 : From the corollaries 1, 2 and 3 it is clear that quasi, dedekind semi-prime Goldie rings, multiplicative semi-prime Goldie rings and hereditary semi-prime Goldie rings are rings with S. P.

Theorem 6—Let M be a singular R-module. Over a semi-prime Goldie ring R and N any proper module extension of M. Then N is singular if and only if for every $x \in N \setminus M$ there exists a regular element R such that $\alpha x \in M$.

PROOF : Let the condition be satisfied. Therefore for every $x \in N \setminus M$ there exists a regular element $\alpha \in R$ such that $\alpha x \in M$. If $\alpha x = 0$ then $\alpha \in O(x)$. Therefore $O(x)$ is an essential ideal of R (Goldie 1960). If $\alpha x \neq 0$ then $O(\alpha x)$ is an essential ideal of R since M is singular. R semi-prime Goldie ring implies that $O(\alpha x)$ contains a regular element say γ (Singh 1968). Now I_α is an essential ideal of R, since it contains a regular element γ_α (Gilmer 1972). But $I_\alpha \subseteq O(x)$ therefore $O(x)$ is an essential ideal of R and hence N is singular.

Conversely, suppose N is a singular R-module. Then for any $x \in N \setminus M$. $O(x)$ is essential in R, therefore it contains a regular element say α and $\alpha x = 0 \in M$, therefore the condition is satisfied.

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