COMPARATIVE OSCILLATION OF SECOND ORDER RETARDED DIFFERENTIAL EQUATIONS

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The nonlinear retarded equation

$$(r(t)\, y'(t))' + a(t)\, y(g(t)) = f(t)$$

is studied for its oscillatory behaviour by comparison with

$$y''(t) + a(t)\, y(g(t)) = 0.$$

It is shown that under certain conditions, all bounded oscillatory solutions of this equation approach zero asymptotically.

INTRODUCTION

Keener (1971) showed that if $h(t)<0$ and continuous on $[0, \infty]$, then

$$h''(t)+a(t)h(t)\geqslant 0 \qquad \ldots(1)$$

is a necessary and sufficient condition for

$$y''(t)+a(t)y(t)=0$$

to be non-oscillatory; where $a(t)>0$ continuous on $[0, \infty]$. Dahiya and Singh (1975a) extended this result to a general retarded equation

$$y^{(2n)}(t)+p(t)\, g(y_\tau(t), y'_{\sigma_1}(t), \ldots, y^{(2n-1)}_{\sigma_{2n-1}}(t))=0 \qquad \ldots(2)$$

$$y_\tau(t)\equiv y(t-\tau(t)),\ y^{(i)}_{\sigma_i}(t)=y^{(i)}(t-\sigma_i(t)),\ y^{(i)}(t)=\frac{d^i}{dt^i}(y(t))$$

$i = 1, 2, \ldots, 2n-1$, by a similar comparative study.

Our purpose here is to compare the oscillatory behaviour of the equations

$$(r(t)\, y'(t))' + a(t)\, y(g(t) = f(t) \qquad \ldots(3)$$

and

$$y''(t) + a(t)\, y(g(t)) = 0 \qquad \ldots(4)$$

where it is assumed for the rest of this note:

(i) $a(t)$, $r(t)$, $g(t)$ are non-negative, continuous on $(-\infty, \infty)$;

(ii) $g(t) \leqslant t$, $g(t) \to \infty$ as $t \to \infty$;

(iii) $f(t) \in C(-\infty, \infty)$.

We call a function $p(t) \in (t_0, \infty)$, $t_0 > 0$ oscillatory if $p(t)$ has arbitrarily large zeros. Otherwise, $p(t)$ is called non-oscillatory.

In what follows, conditions will be found to the effect that if (3) has an oscillatory solution that does not have a limit zero at $t \to \infty$, then bounded solutions of eqn. (4) are oscillatory.

Thus whereas results of Dahiya and Singh (1975a, b), Keener (1971) and Singh (1973) pertain to non-oscillatory behaviour, the results in this note relate to oscillatory nature of two equations being compared. For more on oscillation, the reader is referred to Dahiya and Singh (1973) and Dahiya (1975).

Main Result

Theorem 1 — Suppose

(A) $\int^{\infty} 1/r(t)\, dt < \infty$,

(B) $\int^{\infty} | f(t) |\, dt < \infty$,

(C) equation (3) has bounded oscillatory solution $y(t)$ such that

$\lim_{t \to \infty} \sup y(t) = \alpha \neq 0$;

then equation (4) has all its solutions oscillatory.

Proof : Let T be large enough so that

$$M_1 = \sup | y(t) | > \alpha/2 \qquad \ldots(5)$$

for $t \geqslant T$; and

$$\int_T^{\infty} 1/r(t) dt < 4/\alpha \qquad \ldots(6)$$

Let $T < t_1 < t_2$ be two consecutive zeros of $y(t)$. Let

$$M = \max_{t_1 \leqslant t \leqslant t_2} | y(t) | \text{ ; also } M = y(t_0),\ t_0 \in [t_1, t_2]. \qquad \ldots(7)$$

Now let without any loss $y(t) > 0$ in (t_1, t_2). Then

$$M = \int_{t_1}^{t_0} y'(t)\,dt$$

and

$$M \leqslant \int_{t_1}^{t_0} |\, y'(t)\,|\; dt. \qquad \ldots(8)$$

Similarly

$$M \leqslant \int_{t_0}^{t_2} |\, y'(t)\,|\; dt. \qquad \ldots(9)$$

Adding (8) and (9)

$$2M \leqslant \int_{t_1}^{t_2} |\, y'(t)\,|\; dt.$$

Suppose T is large enough so that

$$\int_{t_1}^{t_2} |\, f(t)\,|\; dt < \frac{k\alpha^2}{4},\, k = \frac{\alpha^2}{4M_1}. \qquad \ldots(9a)$$

Now

$$2M \leqslant \int_{t_1}^{t_2} \frac{1}{\sqrt{r(t)}} . \sqrt{r(t)}\; \sqrt{|\, y'(t)\,|}\; \sqrt{|\, y'(t)\,|}\; dt. \qquad \ldots(10)$$

Applying Schwarz's inequality, we have

$$4M^2 \leqslant \int_{t_1}^{t_2} \frac{1}{r(t)}\, dt \int_{t_1}^{t_2} (r(t)\, y'(t))\, y'(t)\, dt. \qquad \ldots(11)$$

From (5)

$$M^2 > \frac{\alpha^2}{4}; \qquad \ldots(12)$$

and from (6)

$$\int_{t_1}^{t_2} 1/r(t)\; dt < 4/\alpha. \qquad \ldots(13)$$

Substituting from (12) and (13) in (11) we have

$$\frac{\alpha^3}{4} \leqslant \int_{t_1}^{t_2} (r(t)\, y'(t))\, y'(t)\, dt. \qquad \ldots(14)$$

Integrating right-hand side of (14) by parts we have

$$\frac{\alpha^3}{4} \leqslant - \int_{t_1}^{t_2} (r(t)\, y'(t))'\, y(t)\, dt \qquad \ldots(15)$$

since t_1 and t_2 are zeros of $y(t)$. Making use of equation (3) in (15) we have

$$\left.\begin{aligned} \frac{\alpha^3}{4} &\leqslant \int_{t_1}^{t_2} y(t)\, a(t)\, y(g(t)) - \int_{t_1}^{t_2} y(t) f(t)\, dt \\ \frac{\alpha^3}{4} &\leqslant \int_{t_1}^{t_2} y(t)\, a(t)\, |\, y(g(t))\, |\, dt + \int_{t_1}^{t_2} y(t)\, |\, f(t)\, |\, dt \end{aligned}\right\} \qquad \ldots(16)$$

$$\frac{\alpha^3}{4MM_1} \leqslant \int_{t_1}^{t_2} a(t)\, dt + \frac{\int_{t_1}^{t_2} |\, f(t)\, |\, dt}{M_1}. \qquad \ldots(17)$$

Since $M \geqslant y(t)$ and $M_1 \geqslant |\, y(g(t))\, |$

$$\frac{\alpha . \alpha^2}{4M_1^2} \leqslant \int_{t_1}^{t_2} a(t)\, dt + 2/\alpha \int_{t_1}^{t_2} |\, f(t)\, |\, dt, \; M_1 \geqslant M.$$

$$k\alpha \leqslant \int_{t_1}^{t_2} a(t)\, dt + 2/\alpha \int_{t_1}^{t_2} |\, f(t)\, |\, dt \qquad \ldots(18)$$

Since $4M_1^2 > \alpha^2$ and $0 < k < 1$.

Thus

$$\int_{t_1}^{t_2} a(t)\, dt > \frac{k\alpha}{2} > 0, \text{ from } (9a). \qquad \ldots(19)$$

Thus between any two consecutive zeros of $y(t)$, (19) holds.

Hence

$$\int_T^{\infty} a(t)\, dt = \infty. \qquad \ldots(20)$$

But (20) is a sufficient condition for eqn. (4) to be oscillatory.

This completes the proof of the theorem.

Remarks 1 : Consider the equation

$$y''(t) + e^{-t}\, y(g(t)) = 0. \qquad \ldots(21)$$

By Theorem 3 of Singh (1973), eqn. (21) has a non-oscillatory solution.

Hence all bounded oscillatory solutions of

$$(r(t)\, y'(t))' + e^{-t}\, y(g(t)) = f(t)$$

approach to zero as $t \to \infty$ for $r(t)$ and $f(t)$ satisfying (A) and (B) of this theorem. As a consequence all bounded oscillatory solutions of

$$(e^t\, y'(t))' + e^{-t}\, y(g(t)) = \frac{\sin t}{t},\ t > 0 \qquad \ldots(22)$$

approach to zero as $t \to \infty$.

Example 1 — Consider the equation

$$(e^t y'(t))' + e^{-t-2\pi}\, y(t-\pi) = e^{-3t} + 2e^{-t}. \qquad \ldots(23)$$

Since

$$y''(t) + e^{-t-2\pi}\, y(t-\pi) = 0 \qquad \ldots(24)$$

has a non-oscillatory solution, it follows from Theorem 1 that all bounded oscillatory solutions of (23) approach to zero as $t \to \infty$.

In fact $y = e^{-2t}$ is a non-oscillatory solution of (23) that approaches to zero as $t \to \infty$.

The above remark and example lead to following corollary :

Corollary 1 — Suppose eqn. (4) has a non-oscillatory solution. Then all bounded oscillatory solutions of eqn. (3) approach zero asymptotically.

Theorem 2 — Suppose there exists a bounded $h(t) \in C^2(-\infty, \infty)$ such that $h(t)$ is eventually negative and satisfies

$$h''(t) + a(t)\, h(g(t)) \geqslant 0.$$

Then all bounded oscillatory solutions of eqn. (3) approach to zero as $t \to \infty$.

PROOF: By the main theorem of Dahiya and Singh (1975*b*), the conditions here imply

$$\int^{\infty} ta(t)\, dt < \infty \qquad \ldots(25)$$

which in turn implies that (4) has a bounded oscillatory solution. The conclusion now follows by the corollary.

The proof of Theorem 2 is now complete.

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