ON THE ABSOLUTE CÈSARO SUMMABILITY OF INFINITE SERIES

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A series $\sum a_n$ is said to be summable $| C, \alpha |_k$ $(a > -1, k \geqslant 1)$ if

$$\sum n^{k-1} | \sigma_n^\alpha - \sigma_{n-1}^\alpha |^k < \infty$$

where σ_n^α is the nth Cèsaro mean of order α of the series $\sum a_n$. In this note the following theorem has been established.

Theorem — If (λ_n) is a convex sequence such that $\sum \lambda_n/n < \infty$

and $\sum_1^n \frac{| s_v |^k}{v} = O(\log n \gamma_n),\ n \to \infty.$

where (γ_n) is a positive non-decreasing sequence such that

$n \gamma_n \log n \triangle(1/\gamma_n) = O(1)$, then $\sum \frac{a_n \lambda_n}{\gamma_n}$ is summable $| C, 1 |_k$ $(k \geqslant 1)$.

§1. *Definitions* — Let $\Sigma\, a_n$ be a given infinite series with partial sums s_n. Let σ_n^α denote the nth Cèsaro means of order α $(\alpha > -1)$ of the sequence $\{s_n\}$. The series $\Sigma\, a_n$ is said to be absolutely summable (C, α) with index k or simply $| C, \alpha |_k$ if

$$\Sigma\, n^{k-1} | \sigma_n^\alpha - \sigma_{n-1}^\alpha |^k < \infty.$$

The following is due to Mazhar* (1966).

Theorem A — If $\{\lambda_n\}$ is a convex sequence such that

$$\sum \frac{\lambda_n}{n} < \infty \text{ and } \sum_1^n \frac{| s_v |^k}{v} = O(\log n),\ n \to \infty\ (k \geqslant 1)$$

then

$$\sum a_n \lambda_n \text{ is summable } | C, 1 |_k.$$

*The same result was also obtained by Mishra (1965).

§2. In this paper we obtain the generalization of Theorem A. We prove the following theorem :

Theorem — If $\{\lambda_n\}$ is a convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$ and

$$\sum_{1}^{n} \frac{|s_v|^k}{v} = O(\log n \, \gamma_n), \; n \to \infty \qquad \ldots(2.1)$$

where $\{\gamma_n\}$ is a positive non-decreasing sequence and

$$n\gamma_n \log n . \Delta \frac{1}{\gamma_n} = O(1) \qquad \ldots(2.2)$$

then

$$\sum \frac{a_n \lambda_n}{\gamma_n} \text{ is summable } |C, 1|_k, \; k \geqslant 1.$$

It may be remarked that our theorem also includes the following theorem of Singh (1968).

Theorem B — If $\{\epsilon_n\}$ is a convex sequence such that

$$\sum n^{-1} \epsilon_n < \infty, \text{ and}$$

$$\sum_{v=1}^{n} \frac{|s_v|}{v} = O(\log n \, \gamma_n), \; n \to \infty,$$

where $\{\gamma_n\}$ is a positive non-decreasing sequence such that

$$n\gamma_n \log n \, \Delta \, (1/\gamma_n) = O(1), \; n \to \infty$$

then

$$\sum \frac{a_n \epsilon_n}{\gamma_n} \text{ is summable } |C, 1|.$$

§3. For proof of our theorem we require the following lemmas :

Lemma 1 — If $\{\lambda_n\}$ is a convex sequence such that

$\sum \frac{\lambda_n}{n} < \infty$, then $\{\lambda_n\}$ is a non-negative decreasing sequence

and $\lambda_n \log n = O(1), \; n \to \infty$.

Lemma 2 — Under the conditions of Lemma 1, we have

(i) $\sum_{1}^{m} \log (n+1) \triangle \lambda_n = O(1)$

(ii) $m \log (m+1) \triangle \lambda_m = O(1)$

(iii) $\sum_{1}^{m} n \log (n+1) \triangle^2 \lambda_n = O(1), m \to \infty.$

§4. *Proof of the Theorem* — Let T_n denote the nth Cèsaro mean of order 1 of the sequence $\left\{ n a_n \frac{\lambda_n}{\gamma_n} \right\}$. Then we show that

$$\sum_{1}^{\infty} n^{-1} \mid T_n \mid^k < \infty. \qquad \text{...(4.1)}$$

Now

$$T_n = \frac{1}{n+1} \sum_{v=0}^{n} v\, a_v \frac{\lambda_v}{\gamma_v}$$

$$= \frac{1}{n+1} \sum_{1}^{n-1} \triangle \left(\frac{v \lambda_v}{\gamma_v} \right) s_v + \frac{n\, s_n}{n+1} \left(\frac{\lambda_n}{\gamma_n} \right) - \frac{a_0}{n+1} \left(\frac{\lambda_1}{\gamma_1} \right)$$

$$= \frac{1}{n+1} \sum_{1}^{n} \left(\frac{v \lambda_v}{\gamma_v} \right) s_v + s_n \frac{\lambda_{n+1}}{\gamma_{n+1}} - \frac{a_0\, \lambda_1}{(n+1)\gamma_1}$$

$$= L_1^{(n)} + L_2^{(n)} + L_3^{(n)}, \text{ say.}$$

By Minkowski's inequality, it is sufficient to prove that

$$\sum \frac{\mid L_r^{(n)} \mid^k}{n} < \infty \;(r=1, 2, 3,). \qquad \text{...(4.2)}$$

Proof of (4.2) for $r=1$ — We have

$$\sum_{1}^{\infty} \frac{\mid L_1^{(n)} \mid^k}{n} = \sum_{1}^{\infty} \frac{1}{n(n+1)^k} \left| \sum_{1}^{n} \triangle \left(\frac{v\, \lambda_v}{\gamma_v} \right) s_v \right|^k$$

$$\leqslant A^* \sum_{1}^{\infty} \frac{1}{n^{k+1}} \left(\sum_{1}^{n} v \triangle \left(\frac{\lambda_v}{\gamma_v} \right) \mid s_v \mid \right)^k$$

$$+ A \sum_{1}^{\infty} \frac{1}{n^{k+1}} \left(\sum_{1}^{n} \frac{\lambda_{v+1}}{\gamma_{v+1}} \mid s_v \mid \right)^k$$

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*A is a constant not necessarily the same at each occurrence.

$$\leqslant A\left\{\sum_{1}^{\infty}\frac{1}{n^{k+1}}\sum_{1}^{n}\nu\triangle\left(\frac{\lambda_v}{\gamma_v}\right)|\, s_v\,|^{k}\left(\sum_{1}^{n}\nu\triangle\left(\frac{\lambda_v}{\gamma_v}\right)^{k/k'}\right)\right\}$$

$$+\,A\left\{\sum_{1}^{\infty}\frac{1}{n^{k+1}}\sum_{1}^{n}\frac{\lambda_{v+1}}{\gamma_{v+1}}\,|\, s_v\,|^{k}\left(\sum_{1}^{n}\frac{\lambda_{v+1}}{\gamma_{v+1}}\right)^{k/k'}\right\}.$$

$$= O\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{\nu=1}^{n}\nu\triangle\left(\frac{\lambda_v}{\gamma_v}\right)|\, s_v\,|^{k}\right.$$

$$\left.+\,O\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{\nu=1}^{n}\frac{\lambda_v}{\gamma_v}\,|\, s_v\,|^{k}\left(\sum_{1}^{\infty}\frac{\lambda_v}{\nu\gamma_v}\right)^{k/k'}\right.\right),$$

$$= O\left(\sum_{\nu=1}^{\infty}\triangle\left(\frac{\lambda_v}{\gamma_v}\right)|\, s_v\,|^{k}+O\left(\sum_{\nu=1}^{\infty}\frac{\lambda_v}{\nu\gamma_v}\,|\, s_v\,|^{k}\right)\right).$$

Now consider

$$\sum_{\nu=1}^{m}\frac{\lambda_v}{\nu\gamma_v}\,|\, s_v\,|^{k}=\sum_{\nu=1}^{m-1}\triangle\left(\frac{\lambda_v}{\gamma_v}\right)\sum_{p=1}^{\nu}\frac{|\, s_p\,|^{k}}{p}+\frac{\lambda_m}{\gamma_m}\sum_{1}^{m}\frac{|\, s_p\,|^{k}}{p},$$

$$= O\left(\sum_{\nu=1}^{m-1}\triangle\left(\frac{\lambda_v}{\gamma_v}\right)\gamma_v\log\nu\right)+O\left(\frac{\lambda_m\,\gamma_m\log m}{\gamma_m}\right)$$

$$= O(1),\text{ as } m\to\infty.$$

Also

$$\sum_{\nu=1}^{m}\triangle\left(\frac{\lambda_v}{\gamma_v}\right)|\, s_v\,|^{k}=\sum_{\nu=1}^{m}\nu\,\frac{\triangle\lambda_v}{\gamma_v}\,\frac{|\, s_v\,|^{k}}{\nu}+\sum_{\nu=1}^{m}\nu\lambda_{v+1}\triangle\left(\frac{1}{\gamma_v}\right)\frac{|\, s_v\,|^{k}}{\nu},$$

$$=\sum_{\nu=1}^{m-1}\triangle\left(\frac{\nu\triangle\lambda_v}{\gamma_v}\right)\sum_{\mu=1}^{\nu}\frac{|\, s_\mu\,|^{k}}{\mu}+\frac{m\triangle\lambda_m}{\gamma_m}\sum_{\mu=1}^{m}\frac{|\, s\mu\,|^{k}}{\mu}$$

$$+\sum_{\nu=1}^{m}\frac{\nu}{\gamma_v}\,\lambda_{v+1}\triangle\left(\frac{1}{\gamma_v}\right)\frac{|\, s_v\,|^{k}}{\nu}\log\nu\,\gamma_v\,.$$

$$= O\left[\left\{\left(\sum_{\nu=1}^{m-1} \frac{\nu\Delta^2 \lambda_\nu}{\gamma_\nu} + \frac{\Delta\lambda_{\nu+1}}{\lambda_\nu} + \Delta\lambda_{\nu+1}(\nu+1)\Delta\frac{1}{\gamma_\nu}\right) \log \nu\, \gamma_\nu\right\}\right.$$

$$\left. + \left\{\frac{m\Delta\, \lambda_m}{\gamma_m} \gamma_m \log m \; + \left(2 \sum_{\nu=1}^{m} \frac{|\, s_\nu \,|^k \lambda_{\nu+1}}{\nu \log \nu\, \gamma_\nu}\right)\right\}\right]$$

$$= O\,(1), \text{ as } m \to \infty.$$

Hence

$$\sum_1^\infty \frac{|\, L_1^{(n)} \,|^k}{n} < \infty.$$

PROOF OF (4.2) FOR $r=2$ — We have

$$\sum_1^m \frac{|\, L_2^{(n)} \,|^k}{n} = \sum_1^m \frac{1}{n} \left|\, s_n \frac{\lambda_{n+1}}{\gamma_{n+1}} \right|^k$$

$$\leqslant \sum_1^m \frac{\lambda_n^k}{\gamma_n^k} \frac{|\, s_n \,|^k}{n}$$

$$\leqslant A \sum_1^m \frac{\lambda_n}{\gamma_n} \frac{|\, s_n \,|^k}{n}$$

$$= O\left(\sum_1^{m-1} \Delta\left(\frac{\lambda_n}{\gamma_n}\right) \sum_{p=1}^{n} \frac{|\, s_p \,|^k}{p} + \frac{\lambda_m}{\gamma_m} \sum_1^m \frac{|\, s_p \,|^k}{p}\right)$$

$$= O\left(\sum_1^{m-1} \Delta\left(\frac{\lambda_n}{\gamma_n}\right) \gamma_n \log n + \left(\frac{\lambda_m}{\gamma_m} \gamma_m \log m\right)\right)$$

$$= O\,(1).$$

Finally it is clear that

$$\sum \frac{|\, L_3^{(n)} \,|^k}{n} \leqslant A \sum \frac{1}{n^{k+1}}$$

$$< \infty.$$

This completes the proof of the theorem.

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