BOUNDS FOR BURST-ERROR AND RANDOM-ERROR CORRECTING LINEAR CODES

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The paper presents upper bounds on the sufficient number of parity-check digits for linear codes with the following properties : (1) that have minimum weight at least w and correct all bursts of length b or less, and (2) that correct all random errors of weight m or less and all bursts of length b or less.

It is shown that codes in (1) do not necessarily correct a minimum number of random errors while codes in (2) possess a minimum weight property. An example of a code that corrects triple-adjacent and all single and double errors is also discussed.

1. INTRODUCTION

Burst-error-correcting linear codes are suitable for correcting errors which do not occur independently but are clustered over a given length in a code message. However, when we concentrate on the burst error correction alone, the code fails to correct even a minimum number of random errors when these are not in a burst of specified length. In actual communication, while it is all important to consider correction of clustered errors, care must be taken to correct at least upto a specific number of errors no matter where they lie, i.e., burst and random error correction should be handled simultaneously.

Although some codes (e.g. BCH and Reed-Solomon codes) possess the property of simultaneous correction of both types of errors but to the best of our knowledge, this problem still awaits a systematic study. It may appear at the first instance that this requires a suitable minimum weight constraint over the burst-error-correcting code but this is not enough as would be seen after close examination of the situations. Earlier, Sharma and Dass (1974) have studied codes with minimum weight at least w that have no burst of length b or less as a code word and derived an extension of 'Varshamov-Gilbert bound' for that case. While this ensures only burst and random error detection, the present study extends to correction.

In this communication, we have considered the method of constructing codes with following properties :

(1) that have a minimum weight at least w and correct all bursts of length b or less, and

(2) that correct all random errors of weight m or less and all bursts of length b or less ($m < b$).

It would be noted that codes in (1) with a minimum weight condition cannot always be interpreted for correcting a given number of random errors while codes in (2) turn out in a natural way with a minimum weight condition. This has been illustrated with the help of examples. Bounds on the minimum number of parity check digits in the above two situations are obtained for linear codes. These follow from a method of construction similar to that used in obtaining Varshamov-Gilbert bound by Sacks (1958). An example of a triple-adjacent and all single and double error correcting binary code has also been given.

In what follows, a linear code will be taken as a subspace of the space of all n-tuples over $GF(q)$ and a burst of length b will be a vector whose non-zero components are confined to the consecutive b positions, the first and the last of which are non-zero. Weight of a vector will stand for the number of non-zero entries in it (Hamming 1950).

2. Codes with Minimum Weight Correcting Burst Errors

In this section, we impose a minimum weight condition over the burst error correcting code and in Theorem 1 give an extension of Varshamov-Gilbert bound for a code that corrects all bursts of length b or less. This bound assures the existence of a code that corrects all bursts of length b or less or detects all errors of weight $w-1$ or less.

Theorem 1 — Given two positive integers w and b such that $w \leqslant 2b$, a sufficient condition that there exists an (n, k) linear code, $n > 2b$, with minimum weight at least w that corrects all bursts of length b or less is

$$q^{n-k} > [1+(q-1)]^{(n-1,\, w-2)} + \sum_{\substack{r_1, r_2 :\\ r_1+r_2=w-1}}^{r_1+r_2=2b-1} \left[I(q, n; b, r_2) \binom{b-1}{r_1} (q-1)^{r_1} \right] \quad \ldots(1)$$

where $0 \leqslant r_1 \leqslant b-1, \quad 0 \leqslant r_2 \leqslant b,$

$$I(q, n; b, r) = \begin{cases} \binom{n-b}{r}(q-1)^r, & r = 0, 1 \\ (q-1)^r \sum\limits_{i=r}^{b} \binom{i-2}{r-2}(n-b-i+1), & r \geqslant 2 \end{cases}$$

and $$[1+x]^{(m, r)} = \begin{cases} 0, & r<0 \\ 1, & r=0 \\ 1+\binom{m}{1}x+\ldots+\binom{m}{r}x^r, & 0<r\leqslant m. \end{cases}$$

PROOF : The existence of such a code will be shown by constructing an appropriate $(n-k)\times n$ parity-check matrix for the desired code.

Select a non-zero $(n-k)$-tuple as the first column of the parity-check matrix, H. To add subsequent columns to H appropriately let us suppose that we have chosen $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$. While adding the jth column h_j we must ensure two points : (i) that any $w-1$ columns should be linearly independent and (ii) that h_j should not be a linear combination of the preceding $b-1$ together with any b consecutive from first $j-b$ columns. In other words

$$h_j \neq (a_{i_1} h_{i_1} + a_{i_2} h_{i_2} + \ldots + a_{i_{w-2}} h_{i_{w-2}}) \qquad \ldots(2)$$

and $$h_j \neq (b_{j-b+1}h_{j-b+1} + b_{j-b+2}h_{j-b+2} + \ldots + b_{j-1}h_{j-1}) + (c_t h_t + c_{t+1}h_{t+1} + \ldots + c_{t+b-1}h_{t+b-1}) \qquad \ldots(3)$$

where h_i's are any $w-2$ previous columns and h_t's are any b consecutive columns among $h_1, h_2, \ldots, h_{j-b}$.

As there are $q-1$ non-zero coefficients therefore the number of ways in which coefficients a_i's can be chosen is

$$[1+(q-1)]^{(j-1, w-2)} - 1. \qquad \ldots(4)$$

As all possible linear combinations of $w-2$ or fewer columns are included in (4) therefore the coefficients b_j's and c_t's should be so chosen that at least $w-1$ of these taken together are nonzero. To do this, let us choose r_1 number of b_j's and r_2 number of c_t's such that $r_1+r_2 \geqslant w-1$. (The largest values which r_1 and r_2 can attain are $b-1$ and b respectively.) Now r_1 number of b_j's can be selected in

$$\binom{b-1}{r_1}(q-1)^{r_1} \qquad \ldots(5)$$

ways. To choose r_2 number of c_t's is equivalent to evaluate the number of bursts of length b or less with weight r_2 in a vector of length $j-b$. This can be done in (Sharma and Dass 1974)

$$I(q, j; b, r_2) \qquad \ldots(6)$$

ways, where $I(q, j; b, r_2)$ denotes the expression given in the statement of the theorem. From (4), (5) and (6), the total number of linear combinations to which h_j cannot be equal is

$$(4) + \sum_{\substack{r_1, r_2 : \\ r_1+r_2=w-1}}^{r_1+r_2=2b-1} [(5)\,(6)]. \qquad ...(7)$$

The worst conceivable case would be that each choice of coefficients a_i's, b_i's and c_i's yield a distinct sum. Therefore, a column h_j can be added to H provided that the whole set of $q^{n-k} - 1$ non-zero $(n - k)$-tuples is not exhausted by all these linear combinations i.e. jth column can always be added if

$$q^{n-k} - 1 > (7). \qquad ...(8)$$

But for an (n, k) code to exist, inequality (8) should hold for $j = n$ and thus we shall get (1).

Discussion — The result just obtained has been proved for $w \leqslant 2b$. It may be pointed out that

$$0 \leqslant r_1 \leqslant b-1, \quad 0 \leqslant r_2 \leqslant b \qquad \text{when} \quad w \leqslant b$$

and

$$0 \leqslant r_1 \leqslant b-1, \quad 1 \leqslant r_2 \leqslant b \qquad \text{when} \quad b < w \leqslant 2b.$$

For $w > 2b$, the minimum weight of the code becomes at least $2b + 1$ and then the code is capable of correcting all errors of weight b or less covering in particular the correction of all bursts of length b or less. The term involving joint summation over r_1 and r_2 in (7) vanishes reducing the bound obtained in (1) to Varshamov-Gilbert bound.

Corollary 1 — If we drop the weight consideration imposed over the code i.e. if $w = 1$ then the joint summation in inequality (1) splits into product of two separate terms giving

$$\sum_{r_1=0}^{b-1} \binom{b-1}{r_1} (q-1)^{r_1} = q^{b-1}$$

and

$$\sum_{r_2=0}^{b} I(q, j; b, r_2) = q^{b-1}\,[(q-1)(n-2b+1)+1].$$

Also, in place of expression (4) we get -1.

Then the bound takes the form

$$q^{n-k} > q^{2(b-1)}[(q-1)(n-2b+1)+1]$$

which is a result due to Campopiano given in Theorem 4.10 (Peterson 1961).

Example 1 : We now discuss an example of a binary linear code constructed by the synthesis procedure outlined in Theorem 1 by taking $w = 5$ and $b = 3$.

Consider a (8, 2) linear code with parity-check matrix

$$\left[\begin{array}{c|cc} & 1 & 0 \\ & 0 & 1 \\ I_6 & 0 & 1 \\ & 1 & 0 \\ & 1 & 1 \\ & 1 & 1 \end{array}\right].$$

It can be easily seen that the minimum weight of the code is 5 and the syndromes of patterns which are bursts of length 3 or less are all different.

It is important to note that the codes discussed in this section having minimum weight at least w may not correct all errors of weight $\left[\frac{w-1}{2}\right]^*$ or less along with the bursts to be corrected, though it can correct these separately. Such a situation can be seen from Example 1 in which the minimum weight of the code is 5 and the syndrome of the double error pattern (10010000) and that of (00001110), which is a burst of length 3, is same.

The failure to correct $\left[\frac{w-1}{2}\right]$ or less weight patterns together with the bursts of length b or less arises because while deriving the bound we have not taken care of the fact that the syndromes of these two types of error patterns be different. This study is made in the next section.

3. Codes Correcting Burst and Random Errors

In this section, imposing the random error correction constraint over the burst-error-correcting code, we derive an upper bound on the sufficient number of check digits which assures the existence of a code that corrects all random errors of weight m or less and all bursts of length b or less.

Theorem 2 — Given two positive integers m and b such that $m < b$, a sufficient condition that there exists an (n, k) linear code, $n > 2b$, that corrects all combinations of weight m or less and all bursts of length b or less is

$$q^{n-k} > [1 + (q-1)]^{(n-1,\ 2m-1)} + \sum_{\substack{r_1, r_2 :\\ r_1+r_2=2m}}^{r_1+r_2=b+m-1} \left[K(q, n; b, r_1)\binom{n-1}{r_2}(q-1)^{r_2}\right]$$

$$+ \left[\sum_{i=m}^{b-1}\binom{b-1}{i}(q-1)^i\right]\left[\binom{n-b-1}{m}(q-1)^m + L(q, n; b, m)\right] \qquad \ldots(9)$$

*[x] denotes the largest integer contained in x.

where $0 \leqslant r_1 \leqslant b, \quad 0 \leqslant r_2 \leqslant m-1,$

$$K(q, n; b, r) = \begin{cases} \binom{n-1}{r}(q-1)^r, & r = 0, 1 \\ (q-1)^r \sum_{i=r}^{b} \binom{i-2}{r-2}(n-i), & r \geqslant 2 \end{cases}$$

and

$$L(q, n; b, m) = \begin{cases} q^{b-1}[(q-1)(n-2b+1)+1]-1, & \text{if } m = 0 \\ (q-1)^2 \sum_{i=m+1}^{b} (n-b-i+1) \sum_{j=m-1}^{i-2} \binom{i-2}{j}(q-1)^j, & \text{if } m \neq 0. \end{cases}$$

PROOF: As before, after having selected $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$ of the parity-check matrix H, jth column h_j can be added if it fulfils three requirements laid down below.

As a first requirement, since the code is to correct all combinations of weight m or less, the column h_j to be added should be such that it is not a linear combination of any $2m-1$ or fewer previous columns. Any $2m-1$ or less columns out of $j-1$ can be chosen in

$$[1 + (q-1)]^{(j-1,\ 2m-1)} - 1 \qquad \ldots(10)$$

ways.

Next, since the code along with error patterns of weight m or less is required to correct simultaneously all errors which are bursts of length b or less, the syndrome of any error pattern of weight m or less should not be equal to that of any error pattern which is a burst of length b or less. Inequality (11) below assures that the syndrome of any error pattern of weight m or less is not equal to that of any error pattern which is a burst of length b or less out of j components excepting when the burst includes the last component i.e. jth and the exact m-weight pattern is selected from the first $j-b-1$ components which is now taken care of by inequality (12). Thus, the second requirement on h_j is that

$$h_j \neq (a_s h_s + a_{s+1} h_{s+1} + \ldots + a_{s+b-1} h_{s+b-1}) + (b_{t_1} h_{t_1} + b_{t_2} h_{t_2} + \ldots + b_{t_{m-1}} h_{t_{m-1}}) \qquad \ldots(11)$$

and

$$h_j \neq (c_{j-b+1} h_{j-b+1} + c_{j-b+2} h_{j-b+2} + \ldots + c_{j-1} h_{j-1}) + (d_{i_1} h_{i_1} + d_{i_2} h_{i_2} + \ldots + d_{i_m} h_{i_m}) \qquad \ldots(12)$$

where h_s's are any b consecutive and h_t's are any $m-1$ columns among $h_1, h_2, \ldots, h_{j-1}$ and h_i's are any m columns among $h_1, h_2, \ldots, h_{j-b-1}$ with all non-zero d_i's. Since all possible linear combinations of $2m-1$ or fewer columns are included in (10), we should choose coefficients in (11) and (12) such that at least $2m$ a_s's and b_t's taken together and at least m c_j's are nonzero. In order to do so, choose $r_1 a_s$'s and $r_2 b_t$'s such that $r_1 + r_2 \geqslant 2m$. (The largest values which r_1 and r_2 can attain are b and $m-1$ respectively.) Now, $r_1 a_s$'s, which form a burst of length b or less with weight r_1 in a vector of length $j-1$, can be selected in [refer (6) also]

$$K(q, j; b, r_1) \qquad \ldots(13)$$

ways [where $K(q, j; b, r_1)$ denotes the expression given in the statement of the theorem] and $r_2 b_t$'s in

$$\binom{j-1}{r_2}(q-1)^{r_2} \qquad \ldots(14)$$

ways. Further, at least mc_j's can be chosen in

$$\sum_{i=m}^{b-1} \binom{b-1}{i}(q-1)^i \qquad \ldots(15)$$

ways whereas the number of choices in which all non-zero d_i's can be chosen is

$$\binom{j-b-1}{m}(q-1)^m. \qquad \ldots(16)$$

Lastly, the possibility of same syndrome of any two error patterns each of which is a burst of length b or less is to be excluded. Therefore, the third requirement forces that

$$\begin{aligned} h_j \neq\ & (e_k h_k + e_{k+1} h_{k+1} + \ldots + e_{k+b-1} h_{k+b-1}) \\ & + (f_{j-b+1} h_{j-b+1} + f_{j-b+2} h_{j-b+2} + \ldots + f_{j-1} h_{j-1}) \end{aligned} \qquad \ldots(17)$$

where h_k's are any b consecutive columns among $h_1, h_2, \ldots, h_{j-b}$. Keeping in view the situations considered earlier, it is clear that at least m f_j's together with at least $(m+1)$ e_k's should be taken non-zero. The number of ways in which at least m f_j's can be chosen is already given in expression (15). Also, at least $(m+1)$ e_k's, which form a burst of length b or less having weight $m+1$ or more in a vector of length $j-b$, can be chosen in (refer Sharma and Dass 1974, Theorem 3)

$$L(q, j; b, m) \qquad \ldots(18)$$

ways [where $L(q, j; b, m)$ denotes the expression given in the statement of the theorem].

From (10), (13), (14), (15), (16) and (18), the total number of linear combinations is

$$(10) + \sum_{\substack{r_1, r_2 : \\ r_1+r_2=2m}}^{r_1+r_2=b+m-1} [(13)(14)] + (15)[(16) + (18)]. \quad \ldots(19)$$

Hence, a vector h_j for all choices of coefficients must exist if

$$q^{n-k} - 1 > (19). \quad \ldots(20)$$

Thus, an (n, k) code would exist if inequality (20) is satisfied for $j = n$ and then we shall get (9).

Discussion — The above theorem has been proved for $m < b$. However, if $m \geqslant b$, the burst consideration becomes redundant and the second term in (19) and expression (15) vanish. Then the bound obtained in (9) reduces to Varshamov-Gilbert bound.

Corollary 2 — Divorcing the random error correction constraint i.e. setting $m = 0$, the joint summation in inequality (9) splits into product of two separate terms viz.

$$\sum_{r_1=0}^{b} K(q, n; b, r_1)$$

and

$$\sum_{r_2=0}^{m-1} \binom{n-1}{r_2} (q-1)^{r_2} = [1 + (q-1)]^{(n-1,\ m-1)}$$

$$= [1 + (q-1)]^{(n-1,\ -1)} = 0.$$

Also, for $m = 0$

$$[1 + (q-1)]^{(n-1,\ 2m-1)} = 0,$$

$$\sum_{i=m}^{b-1} \binom{b-1}{i} (q-1)^i = q^{b-1}$$

and

$$\binom{n-b-1}{m} (q-1)^m + L(q, n; b, m) = 1 + L(q, n; b, 0)$$

$$= q^{b-1} [(q-1)(n-2b+1) + 1].$$

Thus, as in Corollary 1, the bound in (9) reduces to the result given in Theorem 4.10 of Peterson (1961) due to Campopiano.

Example 2 : Consider a (8, 2) linear code with parity-check matrix

$$\left[\begin{array}{c|cc} & 1 & 0 \\ & 1 & 1 \\ & 0 & 1 \\ I_6 & 1 & 0 \\ & 1 & 1 \\ & 0 & 1 \end{array}\right].$$

It can be easily verified that the syndromes of all single, double and triple-adjacent patterns are altogether different, therefore, this code can correct all random errors of weight 2 or less and all bursts of length 3 or less i.e. all single, double and triple-adjacent errors. Interestingly, it is found that this code can correct all four-adjacent errors also.

It may be pointed out here that the codes discussed in this section correct all bursts of length b or less and, being capable of correcting all errors of weight m or less also, possess minimum weight at least $2m + 1$. It has already been seen that the converse is not true in general.

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