MIXED BOUNDARY VALUE PROBLEM IN THREE DIMENSIONS

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An harmonic function is sought in a domain bounded by a smooth surface, the function and its normal derivative being prescribed on its contiguous parts. Application of variational principle helps state an analogue of the Dirichlet Principle for this mixed boundary value problem and yields a pair of integral equations of the first kind, for the unkown values of the function and its normal derivative on the two parts, respectively.

Functional Analytic approach using symmetrizing kernels of classical potential theory gives the integral equations quickly. It also establishes that the boundary values of the function corresponds to a continuous moment of a double layer and that of its normal derivative to continuous density of a single layer, on the surface. However, the latter has been expressed in terms of the difference of the corresponding solutions of two integral equations formulated with respect to the regions interior and exterior to the surface.

§1. We seek an harmonic function φ in a domain R bounded by a smooth surface S of class B (Lichtenstein 1918), the function and its normal derivative being prescribed continuous functions f and g on the parts S_1 and S_2 of S $(= S_1 + S_2)$, respectively.

The assumptions on the surface assure the validity of the formulae of Green (Kellogg 1929, pp. 212-18) and the divergence theorem (Kellogg 1929, p. 113). Let φ be defined, continuously differentiable and have continuous partial derivatives of the second order in R, then Green's formula may be written as

$$(\nabla\varphi, \nabla\varphi)_R + (\varphi, \nabla^2\varphi)_R = \left(\varphi, \frac{\partial\varphi}{\partial n}\right)_S$$

where ∇ is the del operator and n is the normal to the surface. The suffixes on round brackets indicate integrals over the region. According to the Dirichlet Principle, the minimum of the Dirichlet integral $(\nabla \varphi, \nabla \varphi)_R \geqslant 0$, if it exists, for functions which assume given continuous values on S, is attained for those that are harmonic in R, the minimum then is $\left(\varphi, \frac{\partial\varphi}{\partial n}\right)_S \geqslant 0$.

Ideas similar to the above are applied to the mixed boundary value problem. For this we define :

$$I(\chi) = (g, \chi)_{S_2} - \left(f, \frac{\partial \chi}{\partial n}\right)_{S_1} \qquad ...(1.1)$$

and certain harmonic functions Φ and Ψ in R which approximate φ, so that we may set

$$\Phi = \varphi + \delta \qquad ...(1.2)$$

and

$$\Psi = \varphi + \epsilon \qquad ...(1.3)$$

where δ and ϵ are small error functions such that $\frac{\partial \delta}{\partial n} = 0$ on S_2 and $\epsilon = 0$ on S_1, i. e. Φ and Ψ satisfy the Neumann and Dirichlet parts of the boundary conditions, respectively.

$I(\varphi)$ has physical significance and is related to the difference of the energy stored in the two portions of the three dimensional regions bounded by S_1, S_2 and the interface plane separating the surface S into two parts.

The following inequality is obvious :

$$I(\varphi) - (\nabla\epsilon, \nabla\epsilon)_R \leqslant I(\varphi) \leqslant I(\varphi) + (\nabla\delta, \nabla\delta)_R.$$

Using the properties of the functions defined above, this may be put in the form

$$F(\Psi) \leqslant I(\varphi) \leqslant E(\Phi) \qquad ...(1.4)$$

where

$$E(\Phi) = (g, \Phi)_{S_2} + \left(\Phi, \frac{\partial \Phi}{\partial n}\right)_{S_1} - 2\left(f, \frac{\partial \Phi}{\partial n}\right)_{S_1} \qquad ...(1.5)$$

and

$$F(\Psi) = 2\,(g, \Psi)_{S_2} - \left(f, \frac{\partial \Psi}{\partial n}\right)_{S_1} - \left(\Psi, \frac{\partial \Psi}{\partial n}\right)_{S_2} \qquad ...(1.6)$$

which is similar to the inequality derived somewhat differently by Bartlett and Noble (1961) for the two dimensional case.

§2. We may here investigate conversely, the properties of the functions Φ and Ψ for which $E(\Phi)$ and $F(\Psi)$ have a minimum and a maximum respectively.

Since the minimum $I(\varphi)$ exists, let

$$\Phi = \varphi + \eta\delta \qquad ...(2.1)$$

be another function satisfying Neumann part of the boundary conditions on S_2 i.e. δ satisfying homogeneous Neumann conditions on S_2 and η being an arbitrary constant. Substituting in (1.5) we have

$$E(\Phi) = I(\varphi) + \eta \left\{ I(\delta) + \left(\delta, \frac{\partial \varphi}{\partial n}\right)_{S_1} \right\} + \eta^2 \left(\delta, \frac{\partial \delta}{\partial n}\right)_{S_1} \qquad \text{...(2.2)}$$

The conditions for this to have a minimum $I(\varphi)$,

$$I(\delta) + \left(\delta, \frac{\partial \varphi}{\partial n}\right)_{S_1} = 0 \qquad \text{...(2.3)}$$

and

$$\left(\delta, \frac{\partial \delta}{\partial n}\right)_{S_1} > 0 \qquad \text{...(2.4)}$$

are compatible with the property that Φ and δ are harmonic in R, as they are equivalent to $\left(\delta, \frac{\partial \Phi}{\partial n}\right)_S = \left(\Phi, \frac{\partial \delta}{\partial n}\right)_S$ and $(\nabla \delta, \nabla \delta)_R > 0$, respectively. This implies that φ is also harmonic in R. Thus we may state the following analogue :

"The minimum of $E(\Phi)$, for functions satisfying the Neumann part of the boundary conditions on S_2, is attained by those that are harmonic in R and that the minimum is $I(\varphi)$."

Similarly in (1.4) the maximum of $F(\Psi)$ exists and if

$$\Psi = \varphi + \eta' \epsilon \qquad \text{...(2.5)}$$

be another function satisfying the Dirichlet part of the boundary conditions on S_1, i.e. ϵ satisfying homogeneous Dirichlet condition on S_1 and η' being an arbitrary constant, then (1.6) becomes

$$F(\Psi) = I(\varphi) + \eta' \left\{ I(\epsilon) - \left(\varphi, \frac{\partial \epsilon}{\partial n}\right)_{S_2} \right\} - \eta'^2 \left(\epsilon, \frac{\partial \epsilon}{\partial n}\right)_{S_2} . \qquad \text{...(2.6)}$$

The conditions for this to have a maximum $I(\varphi)$,

$$I(\epsilon) - \left(\varphi, \frac{\partial \epsilon}{\partial n}\right)_{S_2} = 0 \qquad \text{...(2.7)}$$

and

$$\left(\epsilon, \frac{\partial \epsilon}{\partial n}\right)_{S_2} > 0, \qquad \text{...(2.8)}$$

are compatible with the property that Ψ and ϵ are harmonic in R as they are equivalent to $\left(\epsilon, \frac{\partial \Psi}{\partial n}\right)_S = \left(\Psi, \frac{\partial \epsilon}{\partial n}\right)_S$ and $(\nabla \epsilon, \nabla \epsilon)_R > 0$, respectively. This also implies that φ is harmonic in R. Thus :

"The maximum of $F(\Psi)$ for functions satisfying the Dirichlet part of the boundary conditions on S_1, is attained by those that are harmonic in R and that the maximum is $I(\varphi)$."

If in addition, we suppose that $\delta = 0$ on S_1 and $\frac{\partial \epsilon}{\partial n} = 0$ on S_2, then by (2.3) and (2.7), $I(\delta)$ and $I(\epsilon)$ would vanish. In that case δ and ϵ would both satisfy homogeneous mixed boundary conditions on S which through arguments establishing uniqueness (Kellogg 1929, p. 215) implies that both are identically zero. Thus equality in (1.4) is obtained for $\delta = \epsilon = 0$.

§3. We assume the existence of certain expressions L and M (c.f. Green's functions) depending on two points of S and symmetric in them, such that on S, δ and $\frac{\partial \epsilon}{\partial n}$ are given in terms of $\frac{\partial \delta}{\partial n}$ and ϵ by

$$\delta = \left(L, \frac{\partial \delta}{\partial n} \right)_{S_1} \qquad \ldots(3.1)$$

and

$$\frac{\partial \epsilon}{\partial n} = (M, \epsilon)_{S_2}. \qquad \ldots(3.2)$$

Substituting the value of δ in (2.3), inverting the order of integration because of the symmetric property of L and letting $\frac{\partial \varphi}{\partial n} = \Gamma$ on S_1, we have

$$\left((L, \Gamma)_{S_1} + (L, g)_{S_2} - f, \frac{\partial \delta}{\partial n} \right)_{S_1} = 0. \qquad \ldots(3.3)$$

Since $\frac{\partial \delta}{\partial n} \neq 0$ on S_1 or else $\delta \equiv 0$ in R, it follows that for points on S_1, the exact value Γ of the normal derivative is given by

$$(L, \Gamma)_{S_1} = f - (L, g)_{S_2} \qquad \ldots(3.4)$$

which is an integral equation of the first kind. When $L = 1, f = 0$, since for φ harmonic in R, $\left(1, \frac{\partial \varphi}{\partial n}\right)_S = 0$.

Similarly substituting from (3.2) into (2.7) for $\frac{\partial \epsilon}{\partial n}$, inverting the order of integration because of the symmetric property of M and letting $\varphi = \Theta$ on S_2, it follows directly from symmetry that for points on S_2, the exact value Θ of the function is given by

$$(M, \Theta)_{S_2} = g - (M, f)_{S_1} \qquad \ldots(3.5)$$

which is again an integral equation of the first kind.

In the next section, we shall identify L and M through a functional analytic approach.

§4. Integral equations of the first kind also arise when we solve the Dirichlet and Neumann problems employing single and double layer distributions on the surface, respectively. If the function and its normal derivative are prescribed on the boundary, these integral equations may be written in the form :

$$\varphi = (G, \mu)_S \qquad \ldots(4.1)$$

$$\frac{\partial \varphi}{\partial n} = (D, \gamma)_S \qquad \ldots(4.2)$$

where μ and γ are continuous functions to be determined and correspond to the density and moment of the distributions, respectively. If p, q denote points of S, r_{pq}, the distance between them and n_p, the outward normal to S at p, then $G(p, q) = 1/(2\pi r_{pq})$ is the potential at $p(q)$ of a unit mass at $q(p)$ and is symmetric, while $D(p, q) = \dfrac{\partial^2}{\partial n_p \, \partial n_q} G(p, q)$ represents the normal component of force at $p(q)$ due to a unit normal dipole at $q(p)$ and is likewise symmetric.

It is known (Howland 1968) that G and D are the left and right symmetrizers of the kernel $K\left(= \dfrac{\partial G}{\partial n_p}\right)$ of the Fredholm-Poincarè integral equation. Furthermore, in operator notation, $DG = K^2 - I$, where K^2 is the second iterate of K and I is the identity. The necessary and sufficient conditions that (4.1) and (4.2) admit continuous solutions are given in a paper by Howland (1968), Theorem 2 of which states that if $\left(1, \dfrac{\partial \varphi}{\partial n}\right)_S = 0$, then (4.2) has a solution $\gamma = (G, \mu)_S$, where μ satisfies an integral equation of the second kind

$$(K^2 - I)\, \mu = \frac{\partial \varphi}{\partial n}\,. \qquad \ldots(4.3)$$

Thus if φ is chosen to correspond to continuous moment γ on S, then (4.2) gives the integral equation for exact value Θ of φ for points on S_2,

$$(D, \Theta)_{S_2} = g - (D, f)_{S_1}. \qquad \ldots(4.4)$$

Similarly if continuous density μ is chosen to correspond to the solution of (4.3), then (4.1) or (4.2) may be written as

$$\left(D^{-1}, \frac{\partial \varphi}{\partial n}\right)_S = \varphi$$

which gives the integral equation for exact value Γ of $\dfrac{\partial \varphi}{\partial n}$ for points on S_1

$$(D^{-1}, \Gamma)_{S_1} = f - (D^{-1}, g)_{S_2}. \qquad \ldots(4.5)$$

Here D^{-1} is the inverse of D and is given by an infinite series of symmetric operators $-(G + GK^2 + GK^4...)$. Evidently L and M of the previous section may now be identified with D^{-1} and D respectively so that $LM = D^{-1}D = I$.

§5. If $\mu_\pm$ denote the limiting values of $\frac{\partial\varphi}{\partial n}$ for exterior (positive) and interior (negative) regions of S, then the discontinuity relations (Kellogg 1929, p. 309) are given by

$$(K \mp I)\mu = \mu_\pm. \qquad ...(5.1)$$

Using (4.3) we get the following set of integral equations for $\mu_\pm$,

$$(K \pm I)\mu_\pm = \frac{\partial\varphi}{\partial n} \qquad ...(5.2)$$

Furthermore, analogous to (4.1) and (4.2), integral equations corresponding to the interior and exterior regions may be written as

$$\varphi = L^\pm \mu_\pm \; ; \mu_\pm = M^\pm \gamma \qquad ...(5.3)$$

where

$$L^\pm = D^{-1}(K \pm I) \; ; \quad M^\pm = (K \pm I)\, G^{-1}$$

are symmetric operators with the inverse property

$$L^+M^+ = L^-M^- = I.$$

Since

$$\varphi = L^+ \mu_+ = L^+ M^+ \gamma = \gamma$$

the boundary values of the function may be chosen to correspond to the moment of the double layer.

Also from (4.3) and (5.2) we can easily get

$$\mu = \tfrac{1}{2}(\mu_- - \mu_+). \qquad ...(5.4)$$

which expresses the density of the single layer in terms of the difference of solutions of integral equations (5.2).

Thus the harmonic function sought could be given as the potential of a double layer of moment γ, such that $\gamma = f$ on S_1 and $\gamma = \Theta$ on S_2, obtained from the integral equation (4.4). Alternatively it may be expressed as the potential of a single layer of density $\mu = (K^2 - I)^{-1} \frac{\partial\varphi}{\partial n}$, where $\frac{\partial\varphi}{\partial n} = \Gamma$ on S_1, obtained from the integral equation (4.5), and $\frac{\partial\varphi}{\partial n} = g$ on S_2. Once Γ is known, integral equations (5.2) yield μ_+ and μ_- and consequently (5.4) can also be used to obtain μ.

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