ON CONVEX COMBINATION OF CERTAIN ANALYTIC FUNCTIONS

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Denote by $R_t(\alpha)$ the class of analytic functions $G_t(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disc E such that $G_t(z) = (1-t)z + tf(z), 0 \leqslant t \leqslant 1$, where $f(z)$ satisfies the condition $f(0) = 0, f'(0) = 1, \{ |f(z) - 1| / |f(z) + 1| \} < \alpha,\ 0 < \alpha \leqslant 1$. In this paper by using variational techniques we determine the radius of convexity for functions belonging to the class $R_t(\alpha)$, and also distortion theorems, coefficient estimates and some other properties for functions in $R_t(\alpha)$. All the results obtained are sharp.

1. Introduction

Let $R(\alpha)$ denote the class of functions f analytic in the unit disc $E = \{z : |z| < 1\}$, normalized so that $f(0) = 0,\ f'(0) = 1$ and satisfying in E the condition

$$|f'(z) - 1| / |f'(z) + 1| < \alpha, \qquad 0 < \alpha \leqslant 1. \qquad ...(1.1)$$

Let $R_t(\alpha)$ denote the class of functions G_t analytic in E, defined by

$$G_t(z) = (1-t)z + tf(z); \qquad f \in R(\alpha), 0 \leqslant t \leqslant 1. \qquad ...(1.2)$$

The aim of this note is to obtain the following results for functions belonging to $R_t(\alpha)$: We prove that $R_t(\alpha) \underset{\neq}{\subset} R(\alpha)$, determine the radius of convexity, the area of the image of $|z| < r\ (0 < r \leqslant 1)$ by $G_t(z) \in R_t(\alpha)$, the length of the image of $|z| = r$ and coefficient estimates. All the results obtained are sharp. These generalize some results of MacGregor (1962), Padmanabhan (1970) and Gupta (1972b). For allied problems refer to Trimble (1969) and Gupta (1972a).

2. Theorems

Theorem 1 — If $G_t \in R_t(\alpha)$, $0 \leqslant t \leqslant 1$ then $R_t(\alpha) \underset{\neq}{\subset} R(\alpha)$

Proof : From Gupta (1972a) we get

$$G_t'(z) = 1 + t(f'(z) - 1), \qquad ...(2.0)$$

since Re $\{f(z) + 1\} > 0$, it follows that

$$| G_t'(z) - 1 | / | G_t'(z) + 1 | = | t(f'(z) - 1 | / | 2(1-t) + t(f'(z) + 1) |$$
$$< \frac{| f'(z) - 1 |}{| f'(z) + 1 |} < \alpha.$$

Theorem 2 — The radius of convexity of the class $R_t(\alpha)$ is given by one or other of the following formula

$$r_1 = \frac{-A + \sqrt{A^2 + A\alpha}}{\alpha A}$$

$$r_2 = \left\{\frac{\sqrt{(1-A)(1+\alpha)(1+\alpha+3A-A\alpha)} - (1-A)(1+\alpha)}{2A(1+\alpha)}\right\}^{1/2}$$

where $A = (2t - 1)\alpha$, $(-1 < A \leqslant 1)$. The bounds are sharp.

PROOF : Since $f \in R(\alpha)$, from Padmanabhan (1970) we have

$$f(z) = \{1 - \alpha\omega(z)\}/\{1 + \alpha\omega(z)\} \quad \text{...(2.1)}$$

were $\omega(z)$ is analytic in E and $\omega(0) = 0$, $| \omega(z) | < 1$ in E.

From (2.0) and (2.1) we get

$$G_t(z) = \frac{1 - A\omega(z)}{1 + \alpha\omega(z)} \quad \text{...(2.2)}$$

where $A = (2t - 1)\alpha$, $(-1 < A \leqslant 1)$. Differentiating (2.2) we get

$$1 + z\frac{G_t''(z)}{G_t'(z)} = 1 - \frac{(\alpha + A) z\omega'(z)}{(1 - A\omega(z))(1 + \alpha\omega(z))} \quad \text{...(2.3)}$$

Since ω is analytic in E, $\omega(0) = 0$ and $| \omega(z) | < 1$, from Singh and Goel (1971), we have

$$| z\omega'(z) - \omega(z) | \leqslant \{ | z |^2 - | \omega(z) |^2\} / \{1 - | z |^2\} \quad \text{...(2.4)}$$

From (2.3) and (2.4) we have

$$\text{Re}\left\{1 + \frac{zG_t''(z)}{G_t'(z)}\right\} \geqslant 1 - (\alpha + A)\left\{\text{Re}\left[\frac{\omega(z)}{(1 - A\,\omega(z))(1 + \alpha\omega(z))}\right]\right.$$
$$\left. - \frac{| z |^2 - | \omega(z) |^2}{(1 - | z |^2) | 1 - A\,\omega(z) | | 1 + \alpha\,\omega(z) |}\right\} \quad \text{...(2.5)}$$

Following Singh and Goel (1971), we consider the transformation

$$K(z) = \{1 - A\,\omega(z)\}/\{1 + \alpha\,\omega(z)\}, \quad \text{...(2.6)}$$

which maps the disc $| \omega(z) | \leqslant r$ ($| z | = r$) onto the disc $| K(z) - a | \leqslant d$ where $a = \{1 + \alpha A r^2\} / \{1 - \alpha^2 r^2\}$ and $d = \{(\alpha + A) r\}/\{1 - \alpha^2 r^2\}$.

Therefore from (2.5) and (2.6) we get

$$\operatorname{Re}\left\{1+\frac{z\,G_i''(z)}{G_i'(z)}\right\}\geqslant\frac{2A}{(A+\alpha)}+\frac{1}{(A+\alpha)}\left\{\operatorname{Re}\left[\alpha K(z)-\frac{A}{K(z)}\right]-\frac{r^2\,|\,\alpha K(z)+A\,|^2-|\,1-K(z)\,|^2}{(1-r^2)\,|\,K(z)\,|}\right\}. \quad \ldots(2.7)$$

If $K(z)=a+u+iv$, $R=|\,K(z)\,|$ and denoting the right-hand side of (2.7) by $S(u,v)$ we get

$$S(u,\ v)=\frac{2A}{A+\alpha}+\frac{1}{A+\alpha}\left\{\alpha(a+u)-\frac{A(\alpha+u)}{R^2}-\frac{(d^2-u^2-v^2)(1-\alpha^2r^2)}{R(1-r^2)}\right\}. \quad \ldots(2.8)$$

Differentiating (2.8) with respect to v we get

$$\frac{\partial S(u,v)}{\partial v}=\frac{v}{(A+\alpha)R^4}\,T(u,v)$$

where $T(u,v)=2A(a+u)+\{(d^2-u^2-v^2)R+2R^3\}\{(1-\alpha^2r^2)/(1-r^2)\}$

$$\geqslant 2A(a+u)+2R^3\geqslant 2(a+u)\{(a-d)^2+A\}.$$

If $0\leqslant A\leqslant 1$, then it is clear that $T(u,v)>0$ and if $-1<A<0$ we have

$$T(u,v)\geqslant\frac{2(a+u)}{1+\alpha r^2}\{(1+A)(1+Ar^2)-2Ar(1-\alpha)\}>0.$$

Thus $T(u,v)>0$ for $-1<A\leqslant 1$. Hence $\frac{\partial S}{\partial v}>0$ for $v>0$ and $\frac{\partial S}{\partial v}<0$ for $v<0$ and so the minimum of $S(u,v)$ on every chord $u=$ constant is attained when $v=0$. Therefore the minimum of $S(u,v)$ in the disc $|\,K(z)-a\,|\leqslant d$ is attained on the diameter $v=0$. On putting $v=0$ $(R=a+u)$ in $S(u,v)$ we obtain

$$S(u,0)=L(R)=\frac{2A}{A+\alpha}+\frac{1}{A+\alpha}\left\{R\,\frac{(1+\alpha)(1-\alpha r^2)}{1-r^2}-2a\,\frac{(1-\alpha r^2)}{1-r^2}+\frac{(1-A)(1+Ar^2)}{R\,(1-r^2)}\right\} \quad \ldots(2.9)$$

where $a-d\leqslant R\leqslant a+d$, and so $L'(R)=0$ where

$$R=R_0=\sqrt{\{(1-A)(1+Ar^2)\}/\{(1+\alpha)(1-\alpha r^2)\}} \quad \ldots(2.10)$$

and this minimum equals

$$L(R_0)=\frac{2A}{A+\alpha}+\frac{2}{(A+\alpha)(1-r^2)}\times\{\sqrt{(1+\alpha)(1-A)(1+Ar^2)(1-\alpha r^2)}-a(1-\alpha r^2)\}. \quad \ldots(2.11)$$

It is easy to show that, $R_0 < a + d$. But R_0 may not always be greater than $a - d$. In fact $a - d \gtreqless R$ according as

$$1 - 2r + r^2\{1 + 2(\alpha - A) - \alpha A\} + 2A\alpha r^3 - \alpha A r^4 \gtreqless 0 \qquad \ldots(2.12)$$

Denote the left-hand side of (2.12) by $Q(r)$. An analysis of the equation $Q(r) = 0$ shows that for $0 < \alpha \leqslant 1$ and $-1 < A \leqslant 1$ there is only one root in (0, 1). Denoting this root by r_0, we observe that for $0 < r \leqslant r_0$, $Q(r) \geqslant 0$ and hence $a - d > R_0$. So we consider the case $R_0 \notin [a - d, a + d]$ which corresponds to $0 < r \leqslant r_0$. In this case the value R_0 is inadmissible and so the minimum value of $L(R)$ occurs at $R_1 = a - d$. Using (2.9) this minimum equals

$$L(R_1) = L(a - d) = \frac{2A}{A + \alpha} + \frac{(\alpha - A) - 4A\alpha r + \alpha A r^2 (A - \alpha)}{(A + \alpha)(1 - Ar)(1 + \alpha r)}. \qquad \ldots(2.13)$$

Therefore from (2.11) and (2.12) we get

$$\operatorname{Re}\left\{1 + \frac{z\,G_t''(z)}{G_t'(z)}\right\} \geqslant \begin{cases} \dfrac{2A}{A + \alpha} + \dfrac{2}{(A + \alpha)(1 - r^2)}\Big\{\sqrt{(1-A)(1+\alpha)(1+Ar^2)(1-\alpha r^2)} \\ \qquad -(1-\alpha A r^2)\Big\}, \quad R_0 \geqslant R_1 \\ \dfrac{1 - 2Ar - \alpha A r^2}{(1 - Ar)(1 + \alpha r)}, \quad R_0 \leqslant R_1. \end{cases} \qquad \ldots(2.14)$$

If $R_0 \leqslant R_1$, from (2.14) we have $\operatorname{Re}\left\{1 + \frac{z\,G_t''(z)}{G_t'(z)}\right\} > 0$ provided

$$|z| \leqslant r_1, = \frac{-A + \sqrt{A^2 + \alpha A}}{\alpha A}, \quad 0 \leqslant A \leqslant 1, \qquad \ldots(2.15)$$

and $G_t(z)$ is convex for $|z| < r_1$ for such values of t and α for which $r_1 < r_0$. However if $r_1 > r_0$ we can conclude that G_t is convex for $|z| < r_0$. For $-1 < A < 0$ from (2.14) we notice that $\operatorname{Re}\left\{1 + \frac{z\,G_t''(z)}{G_t'(z)}\right\} > 0$ for all $r < 1$. However $R_0 \leqslant R_1$ only for $r_1 \leqslant r_0$, and $r_0 < 1$. Thus we can conclude that G_t is convex for $r < r_0$. Therefore (2.15) gives the radius of convexity when $0 \leqslant A \leqslant 1$ and $r_1 \leqslant r_0$. The bound is sharp for $f'(z) = \frac{1 - \alpha z}{1 + \alpha z}$ so that $G_t(z) = \frac{1 - Az}{1 + \alpha z}$.

We proceed to find if G_t should be convex for larger domain. To this end consider the interval $r_0 < r < 1$. Again $r > r_0$ corresponds to $R_0 > R_1$ and from (2.15) we get $\operatorname{Re}\left\{1 + \frac{z\,G_t''(z)}{G_t'(z)}\right\} > 0$ provided

$$|z| \leqslant r_2 = \left\{\frac{\sqrt{(1 - A)(1 + \alpha)(1 + 3A + \alpha - A\alpha)} - (1 - A)(1 + \alpha)}{2A(1 + \alpha)}\right\}^{\frac{1}{2}} \qquad \ldots(2.16)$$

and therefore G_t maps $|z| < r_2$ onto a convex domain for such values of t and α for which $R_0 > R$. If $0 \leqslant A \leqslant 1$, $R_0 = R_1$ for those values for which $r_2 = r_1$. Substituting for $r = r_1$ in $Q(r) = 0$ we get

$$4t^2\alpha^2 (\alpha^2 - 2\alpha - 4) + 2t\alpha (2 + 7\alpha + 2\alpha^2 - 2\alpha^3) + (1 - 2\alpha^2 - \alpha^4) = 0 \qquad \text{...(2.17)}$$

An analysis of the above equation shows that, it has only one positive root, say t_0, $t_0 \in (\frac{1}{2}, 1)$ for $\alpha \in \left[\frac{(2 - \sqrt{2})(\sqrt{3} + 1)}{2}, 1\right]$. Thus for a given α in this interval we note that for $t = t_0$, $R_0 = R_1$ and so r_1 and r_2 coincide. And for $\alpha < \frac{1}{2}(2 - \sqrt{2})(\sqrt{3} + 1)$ we get $t_0 > 1$ and $R_0 \neq R_1$. If $-1 < A < 0$, $R_1 = R_0$ for those values of t, α for which $r_2 = r_0$. Again a substitution for $r = r_2$ in $Q(r) = 0$ yields those values of t and α for which $r_2 = r_0$.

To show that the result is sharp, we define the function $G_t(z)$ as follows. Let b be defined by the following

$$\frac{r_2 (r_2 - b)}{1 - r_2 b} = \frac{1 - R_0^*}{\alpha R_0^* + A} \quad \text{where } R_0^{*2} = \frac{(1 - A)(1 + A r_2^2)}{(1 + \alpha)(1 - \alpha r_2^2)} \qquad \text{...(2.18)}$$

It is easy to prove that $\dfrac{1 - R_0^*}{\alpha R_0^* + A} < r_2$. In fact

$$(1 - R_0^*)^2 < r_2 (\alpha R_0^* + A) \text{ for } R_0 > a - d \text{ provided}$$

$$(1 - r_2^2 A^2) - 2R_0^* (1 + A\alpha) r_2^2 + R_2^{*2} (1 - \alpha^2 r_2^2) \leqslant 0,$$

that is,

$$R_0^* \geqslant \frac{1 - A r_2}{1 + \alpha r_2} = (a - d)_{r = r_2} \text{ and so } |b| < 1. \text{ Choose}$$

$$f'(z) = \frac{1 - \alpha\, \omega(z)}{1 + \alpha\, \omega(z)} \qquad \text{...(2.19)}$$

where

$$\omega(z) = \frac{z(z - b)}{1 - zb} = z\, \phi(z), \text{ say} \qquad \text{...(2.20)}$$

then $\phi(z)$ is a bilinear transformation mapping E onto itself. For $\phi'(z)$, we get on simplification

$$\phi'(z) = \frac{1 - (\phi(z))^2}{1 - z^2}\,. \qquad \text{...(2.21)}$$

From (2.20) and (2.21) we conclude that $1 + \frac{z\, G_t''(z)}{G_t'(z)} = 0$ provided

$$[(1 - z^2) - (A + \alpha) z^2] - 2A(z\, \phi(z))\,(1 - z^2) + [(A+\alpha) - A\alpha\,(1 - z^2)]\,(z\, \phi(z))^2 = 0 \qquad \ldots(2.22)$$

We shall verify that (2.22) holds for $z = r_2$. Using (2.18) and (2.21), equation (2.22) on simplification reduces to

$$A r_2^4 (1 + \alpha) + r_2^2 (1 - A)(1 + \alpha) - (1 - A) = 0$$

which is true because r_2 satisfies (2.16) and the proof of Theorem 2 is complete.

3. Distortion Theorems

Let $G_t(z) \in R_t(\alpha)$, then from (2.2) and (2.6), it is known that the function $G_1'(z) = \frac{1 - A\,\omega(z)}{1 + \alpha\,\omega(z)}$ assumes values which lie in the disc with center $a = \frac{1 + \alpha\, A r^2}{1 - \alpha^2 r^2}$ and radius $d = \frac{(\alpha + A)\, r}{1 - \alpha^2 r^2}$. The function $L_t'(z) = \frac{1 + Az}{1 - \alpha z}$ is analytic and univalent for $|z| < r$ and maps $|z| \leqslant r$ onto the same disc center a and radius d. Also $L_t'(0) = G_t'(0) = 1$. Thus $G_t'(z)$ is subordinate to $L_t'(z)$ for $|z| < r$. It is clear that $L_t'(z)$ is the derivative of

$$L_t(z) = -(2t - 1)z - 2t\,\alpha \log(1 - \alpha z) \qquad \ldots(3.1)$$

Therefore as an application of the subordination principle we have the following theorem (The details of proof are omitted) :

Theorem 3 — (a) For $G_t(z) \in R_t(\alpha)$ we have

$$\frac{1 - A\,|z|}{1 + \alpha\,|z|} \leqslant \operatorname{Re} G_t'(z) \leqslant |G_t'(z)| \leqslant \frac{1 + A\,|z|}{1 - \alpha\,|z|}. \qquad \ldots(3.2)$$

Equality is attained in the above for $L_t(z)$ given by (3.1)

(b) The area of the image of $|z| < r$ $(0 < r < 1)$ for functions in $R_t(\alpha)$ is maximal for the function $L_t(z)$.

(c) The length of the image of $|z| = r$ $(0 < r < 1)$ under functions in $R_t(\alpha)$ is maximal for the function $L_t(z)$.

Theorem 4 — For $G_t \in R_t(\alpha)$ we have

$$-(2t - 1)\,|z| + 2t\alpha \log(1 + \alpha\,|z|) \leqslant |G_t(z)| \leqslant -(2t - 1)\,|z| - 2t\alpha \log(1 - \alpha\,|z|).$$

Equality is attained for the function L_t given by (3.1).

PROOF : $G_t(z) = \int_0^z G_t'(v)\, dv = \int_0^r G_t'(se^{i\theta})\, e^{i\theta}\, ds, \mid z \mid = r$

Therefore from (3.2) we have

$$\mid G_t(z) \mid \leqslant \int_0^r \mid G_t'(se^{i\theta}) \mid ds \leqslant \int_0^r \frac{1 + A\, s}{1 - \alpha\, s}\, ds$$

$$= (2t - 1) \mid z \mid -2t\alpha \log (1 - \alpha \mid z \mid).$$

Also

$$\mid G_t(z) \mid \geqslant \int_0^r Re\; G_t'(se^{i\theta})\, ds \geqslant \int_0^r \frac{1 - As}{1 + \alpha s}\, ds$$

$$= -(2t - 1) \mid z \mid + 2t\alpha \log (1 + \alpha \mid z \mid).$$

4. Coefficient Estimates

Theorem 5 — Let $G_t \in R_t(\alpha)$ and let G_t be given by

$$G_t(z) = z + \sum_{k=z}^{\infty} a_k\, z^k \qquad \ldots(4.1)$$

then $\mid a_n \mid \leqslant \frac{2at}{n}$ for $n \geqslant 2$.

The bounds are sharp for functions

$$G_{t,\,0}\,(z) = \int_0^z \frac{1 + A\, x^{n-1}}{1 - \alpha\, x^{n-1}}\, dx \qquad \ldots(4.2)$$

PROOF : For $G_t(z) \in R_t(\alpha)$ we have from (2.0)

$$G_t(z) = \frac{1 - A\, \omega(z)}{1 + \alpha\, \omega(z)}$$

Following Clunie (1959) we set

$$\omega(z) = z + \sum_{k=a}^{\infty} b_k\, z^k \qquad \ldots(4.3)$$

Therefore from (4.1) and (4.3) we have

$$\left[(A+\alpha)+\alpha\sum_{k=2}^{\infty} k\, a_k\, z^{k-1}\right]\left[z+\sum_{k=2}^{\infty} b_k\, z^k\right]=-\sum_{k=2}^{\infty} k\, a_k\, z^k. \quad \text{...(4.4)}$$

Equating corresponding coefficients on both side of (4.4), we observe that the coefficient of z^n on the right-hand side of (4.4) depends only on $a_2, a_3, \ldots\ldots, a_n$ on the left-hand side of (4.4) and hence for $n \geqslant 1$ we can write

$$\left[(A+\alpha)+\alpha\sum_{k=2}^{n} k\, a_k\, z^{k-1}\right]\omega(z)=-\sum_{k=2}^{n+1} k a_k\, z^{k-1}+\sum_{k=n+1}^{\infty} c_k\, z^k$$

where c_k's are some complex numbers. Then since $|\,\omega(z)\,| < 1$ we have

$$\left|(A+\alpha)+\alpha\sum_{k=2}^{n} k\, a_k\, z^{k-1}\right| \geqslant \left|-\sum_{k=2}^{n+1} k a_k\, z^{k-1}+\sum_{k=n+1}^{\infty} c_k\, z^k\right| \quad \text{...(4.5)}$$

Squaring both sides of (4.5) and integrating round $|\,z\,| = r < 1$ we get

$$\left[(A+\alpha)^2+\alpha^2\sum_{k=2}^{n} k^2\,|\,ak\,|^2\, r^{2\,(k-1)}\right]$$

$$\geqslant \sum_{k=2}^{n+1} k^2\,|\,a_k\,|^2\, r^{2(k-1)}+\sum_{k=n+1}^{\infty}|\,c_k\,|^2\, r^{2k} \geqslant \sum_{k=2}^{n+1} k^2\,|\,a_k\,|^2\, r^{2k-2};\ n \geqslant 1,$$

as $r \to 1$, we get

$$(A+\alpha)^2 \geqslant (1-\alpha^2)\sum_{k=2}^{n} k^2\,|\,a_k\,|^2+(n+1)^2\,|\,a_{n+1}\,|^2$$

$$\geqslant (n+1)^2\,|\,a_n+1\,|^2,\ n \geqslant 1$$

that is,

$$|\,a_n\,| \leqslant \frac{2t\alpha}{n},\ n \geqslant 2,$$

consider the function $G_{t,\,0}(z)$ given by (4.2). For this function

$$\left|\frac{G_t{}',{}_0(z)-1}{G_t{}',{}_0(z)+1}\right|=\left|\frac{2\alpha t\, z^{n-1}}{2-2\alpha(1-t)\, z^{n-1}}\right|<\left|\frac{\alpha\, z^{n-1}}{1-\alpha(1-t)\, z^{n-1}}\right|<\alpha$$

for $|\,z\,| < 1$.

Therefore $G_{t,0}(z) \in R_t(\alpha)$. Also for $|z| < 1$, $G_t(z)$ have the expansion

$G_{t,0}(z) = z + \left(\frac{2t\alpha}{n}\right) z^n + \ldots\ldots$ for $|z| < 1$, showing that the estimate is sharp.

Theorem 6 — If $G_t \in R_t(\alpha)$ and if μ is a complex number then

$$\left| a_3 - \mu a_2^2 \right| \leqslant \frac{2t\alpha}{3} \max \left\{ 1, \alpha \left| \frac{3t\mu}{2} - 1 \right| \right\}.$$

The estimate is sharp.

PROOF: We follow a method given by Szynal (1972). Since $G_t \in R_t(\alpha)$ then from (2.1) we have

$$G_t'(z) = \frac{1 + (2t - 1)\,\alpha\,\omega(z)}{1 - \alpha\,\omega(z)}$$

where $\omega(z) = \sum_{k=1}^{\infty} b_k z^k$ is analytic and $|\omega(z)| < 1$ for $z \in E$. Equating the coefficients in (4.6) we get

$$b_1 = \frac{a_2}{\alpha t}, \qquad \ldots(4.7)$$

$$b_2 = \frac{3}{2t\alpha}\left(a_3 - \frac{2a_2^2}{3t}\right) \qquad \ldots(4.8)$$

Since $|b_2| \leqslant 1 - |b_1|^2$, then for every complex number v we have

$$|b_2 - vb_1^2| \leqslant |b_2| + |v|\,|b_1|^2 \leqslant 1 + (|v| - 1)|b_1|^2$$
$$\leqslant \max\{1, |v|\}, \text{ since } |b_1| \leqslant 1. \qquad \ldots(4.9)$$

Equality is attained in (4.9) for functions $\omega(z) = z$ and $\omega(z) = z^2$.
From (4.7), (4.8) and (4.9) we get

$$\left| b_2 - vb_1^2 \right| = \frac{3}{2t\alpha} \left| a_3 - a_2^2 \left(\frac{2}{3t} + \frac{2v}{3\alpha t}\right) \right| \qquad \ldots(4.10)$$

putting $\mu = \frac{2}{3t} + \frac{2v}{3\alpha t}$ we get

$$\left| a_3 - \mu a_2^2 \right| \leqslant \frac{2t\alpha}{3} \max \left\{ 1, \alpha \left| \frac{3t\mu}{2} - 1 \right| \right\} \qquad \ldots(4.11)$$

and since (4.9) is sharp then (4.11) is also sharp.

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REFERENCES

Clunie, J. (1959). On meromorphic Schilicht functions *J. Lond. math. Soc.*, **34**, 215-16.

Gupta, R. S. (1972 a). Radius of convexity of convex sum of univalent functions. *Publ. Math. (DEBRECEN)*, **19**, 39-42.

———— (1972 b). Radius of covexity of a class of univalent functions. *J. Indian math. Soc.*, **35**, 291-96.

MacGregor, T. H. (1962). Functions whose derivative has positive real part. *Trans. Am. math. Soc.*, **194**, 532-37.

Padmanabhan, K. S. (1970). On a certain class of functions whose derivatives have a positive real part in the unit disc. *Ann. Pol. Math.*, **23**, 73-81.

Singh, V., and Goel, R. M. (1971). On radii of convexity and starlikeness of some classes of functions. *J. math. Soc. Japan*, (23) **2**, 323-39.

Szynal, J. (1972). Some remarks on coefficients inequality for α-convex functions. *Bull. Acad. Polon. Sci. Ser. Sci. math. Astr. Phys.*, **20**, 917-19.

Trimble, S. Y. (1969). The convex sum of convex functions. *Math. Z.*, **109**, 112-14.