HOMOTOPY AND TENSOR PRODUCT

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In this paper, some inter-connections between homotopy and tensor product has been discussed in the category of para complexes.

1. INTRODUCTION

Here we introduce the notion of homotopy and tensor product in the category of para complexes and prove some related results.

Definition 1.1 — A para-complex of R-modules is a family (X, Y) equal to $\left\{X_n \xrightarrow{d_n} X_{n-1}, X_n \xrightarrow{q_n} Y_n : n \in Z\right\}$ of pair of R-module homomorphisms such that $q_{n-1}\, d_n\, d_{n+1} = 0$. If (X, Y) is a para-complex, then

$$H_n(X, Y) = \frac{\text{Ker } q_{n-1}\, d_n}{d_{n+1}\,(X_{n+1})}$$ is called the nth parahomology of (X, Y). If

$$A_n = \text{Ker } q_n, \text{ then } H_n(X, Y) = \frac{d_n^{-1}\,(A_{n-1})}{d_{n+1}\,(X_{n+1})} = \frac{C_n}{B_n}$$

Let (X, Y) and (X', Y') be two para complexes. A family

$$f = (f, g) = \left\{\left(X_n \xrightarrow{f_n} X'_n,\ Y_n \xrightarrow{g_n} Y'_n\right) : n \in Z\right\}$$

of pair of R-module homomorphism is called a parachain transformation if

(i) $f_{n-1}\, d_n = d'_n f_n$, (ii) $g_n q_n = q'_n f_n$.

Then paracomplexes together with parachain transformations form a category PR. An equivalence in PR is called parachain equivalence.

Definition 1.2 — Let $f = (f_n, f'_n)$ and $g = (g_n, g'_n)$ be parachain transformations between two para-complexes (X, Y) and (X', Y'). Then f and g will be called para homotopic if there is a family of group homomorphisms $s = \{s_n : Y_n \longrightarrow X'_{n+1}\}$ such that $d'_{n+1}\, s_n q_n + s_{n-1}\, q_{n-1}\, d_n = f_n - g_n$, for every $n \in Z$. s is called parahomotopy between f and g.

Proposition 1.3 — Composites of homotopic parachain transformation are homotopic.

PROOF : Let $f = (f_n, f'_n)$ and $g = (g_n, g'_n)$ be two homotopic parachain transformations from (X, Y) to (X', Y') and $\bar{f}' = (\bar{f}_n, \bar{f}_n')$ and $\bar{g}'$ $(\bar{g}_n, \bar{g}'_n)$ be two homotopic parachain transformations from (X', Y') to (X'', Y''). Then to show that $\bar{f}'f$ and $\bar{g}'g$ are homotopic. Let s be a parahomotopy between f and g and s' be a parahomotopy $\bar{f}'$ and $\bar{g}'$. Then we show that $\bar{f}'_{n+1} s_n + s'_n g'_n$ is a homotopy between $\bar{f}'f$ and $\bar{g}'g$. By definition of homotopy we get,

$$d'_{n+1} s_n q_n + s_{n-1} q_{n-1} d_n = f_n - g_n \quad ...(1)$$

$$d''_{n+1} s'_n q'_n + s'_{n-1} q'_{n-1} d'_n = \bar{f}'_n - \bar{g}'_n. \quad ...(2)$$

(1) and (2) give the following equations :

$$\bar{f}'_n d'_{n+1} s_n q_n + \bar{f}'_n s_{n-1} q_{n-1} d_n = \bar{f}'_n f_n - \bar{f}'_n g_n \quad ...(3)$$

$$d''_{n+1} s'_n q'_n g_n + s'_{n-1} q'_{n-1} d'_n g_n = \bar{f}'_n g_n - \bar{g}'_n g_n. \quad ...(4)$$

Adding (3) and (4) we get,

$$d''_{n+1} (\bar{f}'_{n+1} s_n + s'_n g'_n) q_n + (\bar{f}'_n s_{n-1} + s'_{n-1} g'_{n-1}) q_{n-1} d_n = \bar{f}'_n f_n - \bar{g}'_n g_n$$

if we consider the following :

(i) $q'_n g_n = g'_n q_n$

(ii) $d''_{n+1} \bar{f}'_{n+1} = \bar{f}'_n d'_{n+1}$

(iii) $q'_{n-1} d'_n g_n = g'_{n-1} q_{n-1} d_n$ (definition of parachain transformation)

Remark 1.4 : Let f and g be homotopic parachain transformations from (X, Y) to (X', Y'). Then $H_n(f) = H_n(g) : H_n(X, Y) \to H_n(X', Y')$.

Remark 1.5 : A contraction of a paracomplex (X, Y) is a homotopy between the identity and zeroparachain transformations. A para-complex is called contractible if contraction exists. A contractible para-complex is acyclic i.e. $H_n(X, Y) = 0$.

Definition 1.6 — Let $X = (X, Y) = \{(X_n, Y_n)\}$ and $P = (P, Q) = \{(P_n, Q_n)\}$ be two para-complexes. Define

$$X \otimes P = \left\{ \bigoplus_{p+q=n} X_p \otimes P_q, \bigoplus_{p+q=n} Y_p \otimes Q_q, \bar{d}_n \right\}$$

where $\bar{d}_n$ is defined as follows :

$$\bar{d}_n(x \otimes \bar{p}) = d_p(x) \otimes \bar{p} + (-1)^p x \otimes d_q(\bar{p}) \text{ where}$$

$p + q = n$, $x \otimes \bar{p} \in X_p \otimes P_q$.

Then extend $\bar{d}_n$ by linearity. Also $q_X \otimes q_P(x \otimes \bar{p}) = q_X(x) \otimes q_P(\bar{p})$.

Proposition 1.7 — $X \otimes P$ is a para-complex.

PROOF : $\bar{d}_{n-1}\,\bar{d}_n(x \otimes \bar{p}) = \bar{d}_{n-1}(d_p(x) \otimes \bar{p} + (-1)^p\, x \otimes d_q(\bar{p})$

$$= d_{p+q-1}\,(d_p(x) \otimes \bar{p}) + d_{p+q-1}\,((-1)^p\, x \otimes d_q(\bar{p}))$$

$$= d_{p-1}\,(d_p(x)) \otimes \bar{p} + (-1)^{p-1}\, d_p(x) \otimes d_q\,(\bar{p})$$

$$+ (-1)^p\, d_p(x) \otimes d_q(p) + (-1)^p\,(-1)^p\, x \otimes d_{q-1}\, d_q(p)$$

$$= d_{p-1}\, d_p(x) \otimes p + x \otimes d_{q-1}\, d_q(p) \qquad \ldots(5)$$

Now, $q_X \otimes q_P\,(d_{n-1}\,\bar{d}_n(x \otimes p)) = 0$, in view of (5).

Remark 1.8 : $X \otimes P$ is called the tensor product of X and P.

Remark 1.9 : Tensor product is a covariant bifunctor from the product category of para complexes to the category of para complexes.

PROOF : Let $(f, g), (f', g') : (X, Y) \longrightarrow (X', Y')$ be parachain transformations. Define

$$(f \otimes f')_n(x \otimes x') = f_r(x) \otimes f_s'(x'),\ x \in X_r,\ x' \in X_s',\ r + s = n$$

$$(g \otimes f')_n(y \otimes y') = g_r(y) \otimes g'_s(y'),\ y \in Y_r,\ y \in Y_s',\ r + s = n$$

Consider, $\bar{d}_n(f \otimes f')_n(x \otimes x') = \bar{d}_n(f_r\,(x) \otimes f'_s(x'))$

$$= d_r(f_r\,(x)) \otimes f'_s(x') + (-1)^r f_r(x) \otimes d_s(f'_s(x'))$$

$$= f_{r-1}\, d_r(x) \otimes f'_s(x') + (-1)^r f_r(x) \otimes f'_{s-1}\, d_s(x')$$

$$= (f \otimes f')_{r+s-1}\, \bar{d}_{r+s}(x \otimes x') \qquad (r + s = n) \quad \ldots(6)$$

Also, $(q \otimes q')_n\,(f \otimes f')_n\,(x \otimes x') = (q \otimes q')_n\,(f_r(x) \otimes f'_s(x'))$

$$= q_r\, f_r(x) \otimes q'_s\, f'_s(x')$$

$$= g_r\, r_r(x) \otimes g'_s\, q'_s(x')$$

$$= (g_r \otimes g'_s)\,(q_r(x) \otimes q'_s(x'))$$

$$= (g_r \otimes g'_s)\,(g_r \otimes q'_s)\,(x \otimes x') \qquad \ldots(7)$$

From (6) and (7) $((f \otimes f'), (g \otimes g'))$ is a parachain transformation. Also, tensor product is a covariant bifunctor.

Proposition 1.10 — Tensor product of homotopic parachain transformations is homotopic i.e. if

$(f, g), (f', g') : (X, Y) \longrightarrow (X', Y')$ and $(\bar{f}, \bar{g}), (\bar{f}', \bar{g}')$ are pair of homotopic parachain transformations, then $(f, g) \otimes (\bar{f}, \bar{g})$ is homotopic to $(f', g') \otimes (\bar{f}', \bar{g}')$.

PROOF : Let s be a homotopy between (f, g) and (f', g') and t be homotopy between $(\bar{f}, \bar{g})$ and $(\bar{f}', \bar{g}')$. Then obviously $s \otimes I$ is homotopy between $(f, g) \otimes I$ and $(f', g') \otimes I$ and $I \otimes t$ is a homotopy between $I \otimes (\bar{f}, \bar{g})$ and $I \otimes (\bar{f}', \bar{g}')$. I's are identity parachain transformations. The result, now, follows by Proposition 1.3.

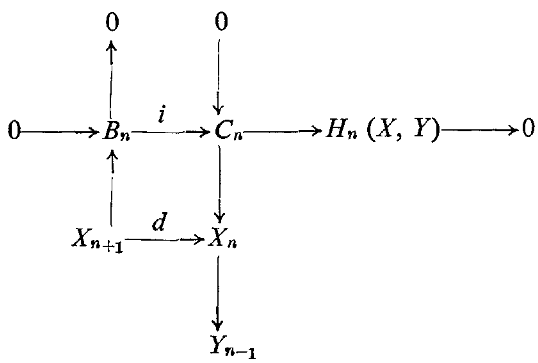
Corollary 1.11 — Tensor product of parachain equivalence is parachain equivalence and tensor product of contractible para complexes, is contractible, hence acyclic.

Corollary 1.12 — We call a parachain transformation $f = (f, f') : (X, Y) \longrightarrow (X', Y')$ para homotopy equivalence if there is another parachain transformation $g = (g, g')$ such that $f'f \simeq I_{(X, Y)}$ and $ff' \simeq I_{(X', Y')}$ where $\simeq$ stands for "homotopic t" and I's stand for identity parachain transformations. Then tensor product of para homotopy equivalence is para homotopy equivalence.

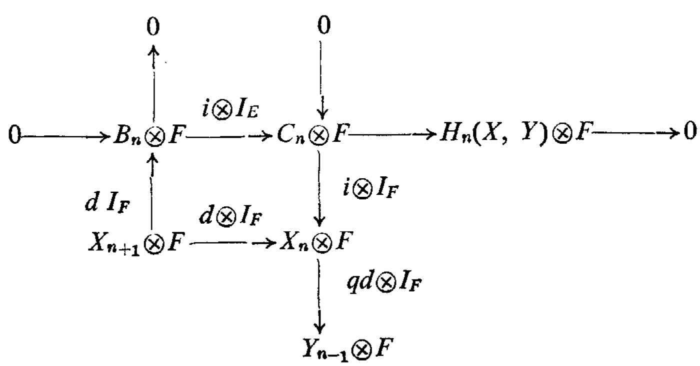
Proposition 1.13 — If F is a flat R-module, then

$$H_n((X, Y) \otimes F) \simeq H_n(X, Y) \otimes F.$$

PROOF: Consider the following commutative diagram with exact rows and columns.



Since F is flat we get the following diagram with extact rows and columns.



From the diagram Ker $qd \otimes I_F = C_n \otimes F$ and $d \otimes I_F (X_{n+1} \otimes F) \simeq B_n \otimes F$.

Hence $H_n(X, Y)\, F \simeq \dfrac{C_n \otimes F}{B_n \otimes F} \simeq H_n(X, Y) \otimes F.$

Corollary 1.14 — If (X, Y) is a para-complex of F-vector spaces V is a F-vector space, then $H_n(X, Y) \otimes V \simeq H_n(X, Y) \otimes V$. In paticular, if

$$H_n((X, Y) \otimes F) \simeq H_n(X, Y) \otimes F.$$

Now we wish to define a covariant functor $J : PC \longrightarrow C$, the category of complex. Let (X, Y) be a para-complex. Define $J(X, Y)_n = \frac{X_n}{d_{n+1}(X_{n+1}) \cap A_n}$, where $A_n =$ Ker q_n . d_n induces $\bar{d}_n : J(X, Y)_n \longrightarrow J(X, Y)_{n-1}$ for

$$d_n(d_{n+1}(X_{n+1}) \cap A_n) \subseteq d_n\, d_{n+1}(A_{n+1}) \cap d_n(A_n)$$
$$\subseteq A_{n-1} \cap d_n(X_n)$$

Also, $q_{n-2}\, d_{n-1}\, d_n(X_n) = 0$ gives $d_{n-1}\, d_n(X_n) \subseteq Ker\, q_{n-2} = A_{n-2}$ and $d_{n-1}\, d_n(X_n) \subseteq d_{n-1}(X_{n-1})$. Hence

$$d_{n-1}\, d_n(X_n) \subseteq A_{n-2} \cap d_{n-1}(X_{n-1}) \text{ i.e. } J(X, Y) \text{ is a complex.}$$

Let $(f, g) : (X, Y) \longrightarrow (X', Y')$ be a para chain transformation. Then we show that $J(f, g) : J(X, Y) \longrightarrow J(X', Y')$ is a chain transformation. For this the following relation must hold

$$f_n(d_{n+1}(X_{n+1}) \cap A_n) \subseteq d'_{n+1}(X'_{n+1}) \cap A'_n \qquad \text{...(A)}$$

By the relation $g_n\, q_n = q'_n\, f_n, f_n(A_n) \subseteq A_n'$ and by $d'_{n+1} f_{n+1} = f_n\, d_{n+1}, f_n\, d_{n+1}(X_{n+1}) \subseteq d'_{n+1}(X'_{n+1})$ is obvious. With these, eqn. (A) is clear.

Thus we can conclude : J is a functor from PR to C. (in fact a covariant functor).

Theorem 1.15 — $H_n J(X, Y) \simeq H_n(X, Y)$.

PROOF : Consider

$$\frac{X_{n+1}}{d_{n+2}(X_{n+2}) \cap A_{n+1}} \xrightarrow{\bar{d}_{n+1}} \frac{X_n}{d_{n+1}(X_{n+1}) \cap A_n} \xrightarrow{\bar{d}_n} \frac{X_{n-1}}{d_n(X_n) \cap A_{n-1}}$$

$$\text{Ker}\, \bar{d}_n = \frac{d_n^{-1}(d_n(X_n) \cap A_{n-1})}{d_{n+1}(X_{n+1}) \cap A_n}$$

$$= \frac{d_n^{-1}\, d_n(X_n) \cap d_n^{-1}(A_{n-1})}{d_{n+1}(X_{n+1}) \cap A_n}$$

$$= \frac{d_n^{-1}(A_{n-1})}{d_{n+1}(X_{n+1}) \cap A_n}$$

$$\text{Image } \bar{d}_{n+1} = \frac{d_{n+1}(X_{n+1}) + d_{n+1}(X_{n+1}) \cap A_n}{d_{n+1}(X_{n+1}) \cap A_n}$$

$$= \frac{d_{n+1}(X_{n+1})}{d_{n+1}(X_{n+1}) \cap A_n}$$

$$\text{i.e.} \quad H_n J(X, Y) \simeq \frac{d_n^{-1}(A_{n-1})}{d_{n+1}(X_{n+1})} \simeq H_n(X, Y).$$

2. Notion of Parahomology Product

Let $X = \{(X_n, Y_n)\}$ and $K = \{(K_n, L_n)\}$ be two para complexes. We define a map $p : H_m(X) \otimes H_q(K) \longrightarrow H_{m+q}(X \otimes K)$ as follows :

Let $x \in C_m^X$, the modules of n-paracycles in X, $k \in C_q^K$, then $x \otimes k \in C_{m+q}^{X \otimes K}$. Now, define $p([x] \otimes [k]) = [x \otimes k]$ where $[x]$, $[k]$ and $[x \otimes k]$ denote the corresponding homology classes. p is a map. For this suppose $[x] = [\bar{x}]$ and $[k] = [\bar{k}]$ Then $(x - \bar{x}) \in B_m^X$, the modules of n-paraboundaries, and $(k - \bar{k}) \in B_m^K$, i.e. $x = \bar{x} + b_m$, $k = \bar{k} + b_q$ where b_m and b_q are paraboundaries. Then

$$x \otimes k = (\bar{x} \otimes \bar{k}) + (b_m \otimes \bar{k}) + (\bar{x} \otimes b_q) + (b_m \otimes b_q).$$

It is, therefore, sufficient to show that tensor product of a paracycle with a paraboundary is a paraboundary in $X \otimes K$. Let $x \in C_m^X$, $k \in B_q^K$. Then $k = d(k')$ for some $k \in K_{q+1}$. Hence $d_{m+q}(x \otimes k') = d(x) \otimes d(k') + (-1)^m X \otimes d(k') = (-1)^m x \otimes k$ i.e. $x \otimes k$ is a paraboundary in $X\ K$. p, now, can be extended to a homomorphism from $\bigoplus_{m+q=n} H_m(X) \otimes H_q(K)$ to $H_n(X \otimes K)$. This homomorphism is called the parahomology product. Our aim is to study it. First of all we have the following which has been proved in Proposition 1.13.

Proposition 2.1 — If G is a flat R-module and (X_n, Y_n) is a para-complex, then $H_n(X \otimes G) \simeq H_n(X) \otimes G$ for each n.

Further, we have to relate tensor, parahomology and functor J.

Theorem 2.2 — If for a para-complex (X, Y), $C_n^{J(X)}$ and $B_n^{J(X)}$ are flat modules for each n. Then the following is exact for any para-complex (K, L).

$$0 \longrightarrow \bigoplus_{m+q=n} H_m(X, Y) \otimes H_q(K, L) \longrightarrow H_n(J(X, Y) \otimes J(K, L)) \longrightarrow$$

$$\bigoplus_{m+q=n-1} \operatorname{Tor}(H_m(X, Y), H_q(K, L)) \longrightarrow 0.$$

Before proving it, we state Kunneth formula. For the proof one may go through Homology by MacLane.

Theorem 2.3 — Let X and K be two complexes such that C_n^X and B_n^X are flat, then for each n, we have an exact sequence.

$$0 \longrightarrow \bigoplus_{m+q=n} H_m(X) \otimes H_q(K) \longrightarrow H_n(X \otimes K) \longrightarrow \bigoplus_{m+q=n-1} \operatorname{Tor}(H_m(X), H_q(K)) \longrightarrow 0$$

PROOF OF THEOREM 2.2 : Under the conditions of the theorem, by Kunneth formula, we get following exact sequence

$$0 \longrightarrow \bigoplus_{m+q=n} H_m J(X, Y) \otimes H_q(J(K, L)) \longrightarrow H_n(J(X, Y) \otimes J(K, L)) \longrightarrow \bigoplus_{m+q=n-1} \operatorname{Tor}[H_m(J(X, Y)), H_q(J(K, L))] \longrightarrow 0$$

which implies the required result if we remember that $H_n(J(X, Y)) \simeq H_n(X, Y)$ and $H_n(J(K, L)) \simeq H_n(K, L)$.

Corollary 2.4 — If for a para-complex (X, Y), $B_n^{J(X)}$ and $H_n(X, Y)$ are flat modules. Then

$$\bigoplus_{m+q=n} H_m(X, Y) \otimes H_q(K, L) \simeq H_n(J(X, Y) \otimes J(K, L)).$$

PROOF : Under these conditions, $C_n^{J(X)}$ is flat for it is direct summand of $H_n(X, Y)$.

Finally we give dual of the Proposition 1.13.

For this we require the definition of pseudocomplex. For further details, one may see Ramji Lal (1974).

Definition 3.1 — A pseudo-complex $(X, A) = (X_n, A_n)$ of R-modules is a family $\left\{X \xrightarrow{d_n} X_{n-1}.\ A_n \text{ is a submodule of } X_n\right\}$ of module-homomorphisms such that $d_{n-1}\, d_n(A_n) = 0$. $H_n(X, A) = \dfrac{\operatorname{Ker} d_{n-1}}{d_{n+1}(A_{n+1})} = \dfrac{C_n}{B_n}$ is called the nth pseudohomology of (X, A).

Remark 3.2 : If (X, A) is a pseudocomplex, then $H_{om}((X, A), C)$ is a para-complex.

Proposition 3.3 — Let (X, A) be a pseudocomplex of R-modules and C an injective R-module. Then $H_n\, H_{om}((X, A), C) \simeq H_{om}(H_n(X, A), C)$, for each n i.e. H_n and functor H_{om} commute.

PROOF : If (X, A) is a pseudocomplex, then $H_n(X, A) = \frac{\text{Ker } d_n}{d_{n+1}(A_{n+1})} = \frac{C_n}{B_n}$.
Functor $H_{om}(\;, C)$ gives the following para-complex.

$$\begin{array}{ccccccc} \longrightarrow H_{om}(X_{n-1}, C) & \xrightarrow{\delta} & H_{om}(X_n, C) & \xrightarrow{\delta} & H_{om}(X_{n+1}, C) & \longrightarrow \\ \downarrow q & & \downarrow q & & \downarrow q & \\ H_{om}(A_{n-1}, C) & & H_{om}(A_n, C) & & H_{om}(A_{n+1}, C) & \end{array}$$

Now $Ker\ q\delta = \{f : X_n \longrightarrow C \text{ such that } f\, d_{n+1}(A_{n+1}) = 0\}$

$$= H_{om}\left(\frac{X_n}{B_n}, C\right)$$

Image $\delta = \{\delta(f) : f \in H_{om}(X_{n-1}, C)\}$

$$= \{f\, d_n : f : X_{n-1} \to C\}$$

$$= H_{om}(B'_{n-1}, C) \text{ where image } d_n = B'_{n-1}$$

$$= H_{om}\left(\frac{X_n}{C_n}, C\right)$$

The sequence $0 \longrightarrow \frac{C_n}{B_n} \longrightarrow \frac{X_n}{B_n} \longrightarrow \frac{X_n}{C_n} \longrightarrow 0$ is exact. The injectivity of C gives following exact sequence

$$0 \longrightarrow H_{om}\left(\frac{X_n}{C_n}, C\right) \longrightarrow H_{om}\left(\frac{X_n}{B_n}, C\right) \longrightarrow H_{om}\left(\frac{C_n}{B_n}, C\right) \longrightarrow 0$$

So,

$$H_{om}\left(\frac{C_n}{B_n}, C\right) \simeq \frac{H_{om}\left(\frac{X_n}{B_n}, C\right)}{H_{om}\left(\frac{X_n}{C_n}, C\right)}.$$

Then, $H_n(H_{om}(X, A), C) \simeq \frac{\text{Ker } q\delta}{\text{Image } \delta}$

$$\simeq \frac{H_{om}\left(\frac{X_n}{B_n}, C\right)}{H_{om}\left(\frac{X_n}{C_n}, C\right)}$$

$$\simeq H_{om}\left(\frac{C_n}{B_n}, C\right)$$

$$\simeq H_{om}(H_n(X, A), C)$$

Remark 3.4 : Let (X, A) be a pseudocomplex of R-modules such that $\frac{X_n}{C_n}$ is projective for each n. Then $H_{om}(H_n(X, A), C) \simeq H_n(H_{om}(X, A), C)$.

Corollary 3.5 — Let (X, A) be a pseudocomplex of F-vector space and V is a F-space. Then $H_{om}(H_n(X, A), V) \simeq H_n(H_{om}(X, A), V)$. In particular $H_{om}(H_n(X, A), F) \simeq H_n(H_{om}(X, A), F)$ i.e. $H_n(H_{om}(X, A), F)$ is dual space of $H_n(X, A)$.

Remark 3.6 : If original pseudocomplex is acyclic, then the para-complex obtained is also acyclic, i.e. the nth parahomology is trivial for each n.

Reference

Ramji Lal (1974). Extensions of groups. (Unpublished).