EXTENSION OF GROUPS

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In this paper the computation of parahomology groups for some particular extensions G of H has been discussed.

INTRODUCTION

The notion of parahomology groups for an extension G of H has been developed (Kumar 1974). In this note, we compute these groups under some restrictions.

1. PRELIMINARIES

Here are preliminaries needed for the discussion (Ramji Lal 1974). Let H be a subgroup of $\{G,.\}$ and S a set of right coset representative system of G modulo H containing identity (call such a system as 'admissible coset representative system' and denote the set of these systems by s). Suppose $x, y \in S$ and $h \in H$. Then $xh = \sigma_x(h)\, x \;\theta\; h$ and $x.y = f(x, y)\, x \;o\; y$, where $\sigma_x(h)$, $f(x, y) \in H$ and $x \;\theta\; h, x \;o\; y \in S$. This gives an action θ of H on S, a map $S \times S \xrightarrow{f} H$. The various identities relating θ, σ, f and o allow us to say that (S, H, σ, f) is a 'C-groupoid' in the following sense :

Definition 1.1 — A system (S, H, σ, f), where S is a groupoid with identity e, H a group acting on S through an action θ, σ a map from S to H^H and f a map from $S \times S$ to H satisfying the following conditions is called a 'C-groupoid' :

(1) $[x \;o\; y = y] \Rightarrow y = e$

(2) For each $x \in S$, there is x' such that $x' \;o\; x = e$.

(3) $\sigma_x(h_1 h_2) = \sigma_x(h_1)\, \sigma_{x\theta h_1}(h_2)$

(4) σ_x is onto for some x.

(5) $f(x, e) = f(e, x) = 1$; 1 is the identity of H.

(6) $(x \;o\; y) \;o\; z = x \;\theta\; f(y, z) \;o\; y \;o\; z$; $x, y, z \in S$.

(7) $(x \;o\; y) \;\theta\; h = x \;\theta\; \sigma_y(h) \;o\; y \;\theta\; h$; $x, y \in S, h \in H$

(8) $f(x, y) f(x \;o\; y, z) = \sigma_x(f(y, z))\, f(x \;\theta\; f(y, z), y \;o\; z)$

(9) $f(x, y)\, \sigma_{xoy}(h) = \sigma_x \sigma_y(h)\, f(x \;\theta\; \sigma_y(h), y \;\theta\; h)$

Thus each S determines a C-groupoid and conversely, the following :

Theorem 1.2 — For a C-groupoid (S, H, σ, f), there is an extension G of H which has S as an admissible coset representative system such that the corresponding C-groupoid is (S, H, σ, f).

Definition 1.3 — A 'C-homomorphism' between $(S^1, H^1, \sigma^1, f^1)$ and $(S^2, H^2, \sigma^2, f^2)$ is triple (p, q, g) where $S_1 \xrightarrow{p} S^2$ is a map, $H^1 \xrightarrow{q} H^2$ is a homomorphism, $S^1 \xrightarrow{g} H^2$ a map such that $g(e) = 1$ satisfying.

(i) $p(x \; o \; y) = p(x) \; \theta \; g(y) \; o \; p(y)$

(ii) $q(f^1(x, y) \, g(x \; o \; y) = g(x) \, \sigma_p(x) \, (g(y)) \, f^2(p(x) \; \theta \; g(y), p(y))$.

(iii) $p(x \; \theta \; h) = p(x) \; \theta \; q(h)$

(iv) $q\left(\sigma^1_x(h) \right) = g(x) \, \sigma^2_{p(x)} \left(q(h) \right) g(x \; \theta \; h)^{-1}$

The above C-homomorphism is called *regular* if $g(x) = 1$, for each $x \in S^1$.

Definition 1.4 — *A factor system* is a C-groupoid in which the action θ is trivial and a C-groupoid is a C-group if S is a group.

Definition 1.5—Let (S, H, σ, f) be a C-groupoid. Consider the set

$$Z(S) = \left\{ S \xrightarrow{g} Z \mid g(x) = 0 \text{ for all but finitely many } x; \; Z \text{ is the set of integers} \right\}$$

Define $+$ and (.) in $Z(S)$ as follows :

$$(f + g)(x) = f(x) + g(x), \; (f. g)(x) = \sum_{y \, o \, z \, = \, x} f(y) \, g(z).$$

Then $Z(S)$ is called the 'integral C-groupoid ring'.

Definition 1.6 — A pseudocomplex (X, Y) of R-modules is a family $\{(X_n, Y_n), X_n \xrightarrow{d_n} X_{n-1}\}$ where Y_n is a submodule of X_n and d_n is a homomorphism such that $d_{n-1} \, d_n(y_n) = 0$. The pseudocomplexes form a category if we define a morphism between (X, Y) and (X', Y') as a family of homomorphisms such that (i) $f_{n-1} \, d_n = d_n' f_n$ (ii) $f_n(Y_n) \subseteq Y_n'$.

Definition 1.7 — Let (S, H, σ, f) be a C-groupoid. A sequence $(x_1, x_2, .., X_n, e, e...)$ is called a σ-sequence if $\sigma_{x_i} \; (f(x_{i+1}, x_{i+2})) = 1$ for $i > 1, x_j \in S$. We define $A_{i_1 \, i_2 \ldots i_r} \, (x_1, x_2, \ldots, x_n, \ldots)$ inductively as follows :

$$A_{i_1}(x_1, x_2, \ldots x_n, \ldots) = x_1, x_2, \ldots, x_{i_1-1}, x_{i_1}\,\theta f(x_{i_1+1}, x_{i_1+2}),$$
$$x_{i_1+1}\; o\; x_{i_1+2}, x_{i_1+3}, \ldots)$$

and supposing $A_{i_1 i_2 \ldots i_r}(x_1, x_2, \ldots, x_n, \ldots) = y_1, y_2, \ldots, y_n, \ldots)$, define

$$A_{i_1 i_2 \ldots i_r}(x_1, x_2, \ldots, x_n, \ldots) = y_1, y_2, \ldots y_{i_r-1}, y_{i_r}\,\theta f(y_{i_r+1}, y_{i_r+2})$$
$$y_{i_r+1}\; o\; y_{i_r+2} \ldots)$$

A σ-sequence is called A-sequence if each $A_{i_1 i_2, \ldots i_r}(x_1, x_2, \ldots, x_n, \ldots)$ is an A-sequence. If $(x_1, x_2, \ldots, x_n, \ldots)$ is an A-sequence, each $A_{i_1 i_2 \ldots i_r}(x_1, x_2, \ldots, x_n, \ldots)$ is an A-sequence.

For the following one may refer to Kumar (1974, 1975c).

Definition 1.8—A family $(X, Y) = \left\{ X_n \xrightarrow{d_n} X_{n-1},\ X_n \xrightarrow{q_n} Y_n \right\}$ of pair of module-homomorphisms is called a para-complex if $q_{n-1}\, d_n d_{n+1} = 0$ $H_n(X, Y)$ is called the nth parahomology of (X, Y). The para-complexes form a category, if we define a morphism between (X, Y) and (X', Y') to be a family $\left\{ X_n \xrightarrow{f_n} X_n',\ Y_n \xrightarrow{g_n} Y_n' \right\}$ of homomorphisms such that

(i) $f_{n-1}\, d_n = d_n'\, f_n$ (ii) $g_n q_n = q_n'\, f_n$.

Let (S, H, σ, f) be a C-groupoid and X_n be the free $Z(S)$-module generated over the elements of S^n *i.e.* $X_n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in S\}$.

Put $Y_n = \{[x_1, x_2, \ldots, x_n] \mid {}_{n-2}\,\theta f(x_{n-1}, x_n) = x_{n-2}, n > 3\}$. Define $X_n \xrightarrow{d_n} X_{n-1}$ as follows :

$$d[x_1, x_2, \ldots, x_n] = [x_2, x_3, \ldots, x_n]$$
$$+ \sum_{t=1}^{n-1} [x_1, x_2, \ldots, x_{t-1}\,\theta f(x_t, x_{t+1}), x_t\; o\; x_{t+1}, x_{t+2}, \ldots, x_n]$$
$$+(-1)^n [x_1, x_2, \ldots, x_{n-1}] \text{ if}$$

$(x_1, x_2, \ldots, x_n, e, e, \ldots)$ is an A-sequence $= 0$ otherwise.

Proposition 1.9—$(X, Y) = (X_n, Y_n)$ is a pseudocomplex.

Thus for a C-groupoid, there is a para-complex $H_{om}((X, Y), C)$ and its nth parahomology, denoted by $H^n(S, H, C)$, is called the nth parahomology of (S, H, σ, f). Let S_H be the category whose objects are C-groupoids with fixed H and morphisms are regular C-homomorphism (p, q, g) with q injective. Then there is a contravariant function T^* from S_H to the category **PR** of para complexes with coefficient in C. The nth parahomology $H^n(,C)$ of Hom $((X, Y), C)$ attached to (S, H, σ, f) is contravariant from S_H to **A**, where **A** denotes the category of abelian groups. We denote by $T^*(S, H, C)$, the paracomplex attached to (S, H, σ, f). Let G be an extension of H and **S** be the set of admissible coset representative systems. Then each $S \in \mathbf{S}$ gives a C-groupoid denoted by (S, H). Define $T^*\left(\frac{G}{H}, C\right) = \bigoplus_{S \varepsilon \mathbf{S}} T^*(S, H)$. Then the nth parahomology, denoted by $H^n\left(\frac{G}{H}, C\right)$, is called the nth parahomology of the extension. It can be remarked that $H^n\left(\frac{G}{H}, C\right) \simeq \bigoplus_{S \varepsilon \mathbf{S}} H^n(S, H, C)$.

Let E_H be the category of extensions of the group H. Then T^* induces a contravariant functor from E_H to **PR**. $H^n(, C)$ is, therefore contravariant from E_H to **A**. Let $(X, Y) = \{(X_n, Y_n)\}$ be the pseudocomplex attached to (S, H, σ, f). Then $(X, Y) = (A, A') \oplus (B, B')$ where

$$A_n = \{[x_1, x_2, \ldots, x_n] \mid (x_1, x_2, \ldots, x_n, e, e, \ldots) \text{ is not an } A\text{-sequence}\}$$

$$B_n = \{[x_1, x_2, \ldots, x_n] \mid (x_1, x_2, \ldots, x_n, e, e, \ldots) \text{ is an } A\text{-sequence}\}$$

$$A_n' = X_n \cap A_n, \quad B_n' = Y_n \cap B_n$$

We shall call A, the pseudocomplex of non A-sequences and B, that of A-sequences.

Proposition 1.10 — $H^n(S, H, C) \simeq \overset{\alpha}{\oplus} C \oplus H^n(B, C)$ where α is cardinality of non-A-sequences $(x_1, x_2, \ldots, x^n, e, e_n \ldots)$ in (S, H, σ, f).

Proposition 1.11—For a C-group (S, H, σ, f), $H^n(S, H, C) \simeq \operatorname{Ext}^n_{Z(S)}(Z, C)$ for $n \leqslant 3$ and hence if H is normal in G, then $H^n\left(\frac{G}{H}, C\right) \simeq \overset{k}{\oplus} \operatorname{Ext}^n_{Z(G/H)}(Z, C)$ for $n \leqslant 3$, where k is the cardinality of **S**.

In computing parahomology groups we take $C = \mathrm{C}(H)$, the centre of H and write $H^n\left(\frac{G}{H}, \mathrm{C}(H)\right) = H^n\left(\frac{G}{H}\right)$.

Proposition 1.12—Let G be a quadratic extension of H. Then for $n > 3$, $H^n(G/H) \simeq \overset{q}{\oplus} C(H) \overset{p}{\oplus} \operatorname{Ext}^n_{Z(Z_2)}(Z, C(H))$ where p is the cardinality of elements of order two in $G - H$, $h = |H|$, $q = 2^n - 4(n-1)(h-p)$.

Proposition 1.13 — Suppose for a C-group (S, H, σ, f), f is trivial. Then $H^n(S, H, C) \simeq \operatorname{Ext}^n_{Z(S)} (Z, C)$ for each n.

2. Computation of $H^n(G/H)$ for some Quadratic Extension

For further discussion we require the following result. For the proof, one may refer to Kumar (1975a).

Theorem 2.1 — Let G be an extension of $H \neq \{0\}$ such that, f, is trivial for each admissible coset representative system. Then the index of H in G is two and each element in G-H is of order two. Also H is abelian.

Theorem 2.2 — Let G be an extension of $H \neq \{0\}$ such that $f(x, y) = 1$ for each admissible coset representative system of G module H. Then

$H^n(G/H) \simeq \overset{|H|}{\oplus} \operatorname{Ext}^n_{Z(Z_2)} (Z, H)$ where $|H|$ is the order H.

Proof : Under the hypothesis, the index of H in G is two, each element in G-H is of order two and H is abelian (Theorem 2.1). Then $H^n(G/H) \simeq \overset{|H|}{\oplus} \operatorname{Ext}^n_{Z(Z_2)} (Z, H)$ (Propositions 1.11 and 1.12).

Theorem 2.3 — Let G be an abelian group which is an extension of $H \neq \{0\}$ such that $f(x, y) = 1$ for each admissible coset representative system of G. Then $H^n(G/H) \simeq \overset{h}{\oplus} H$.

Proof : $G = H \oplus Z_2 \simeq H \oplus \{-1, 1\}$ gives G-$H = \{(h, -1) : h \in Z\}$. Since each element of G-H is of order two, G is a group of exponent two. Thus by Theorem 2.1 and Proposition 1.12

$$H^n(G/H) \simeq \overset{h}{\oplus} \operatorname{Ext}^n_{Z(Z_2)} (Z, \overset{\alpha}{\oplus} Z_2) \simeq \overset{h}{\oplus} \overset{\alpha}{\oplus} \operatorname{Ext}^n_{Z(Z_2)} (Z, Z_2)$$

$$\simeq \overset{h\alpha}{\oplus} Z_2 \simeq \overset{h}{\oplus} H, \text{ where } G \simeq \overset{\alpha+1}{\oplus} Z_2$$

3. $H^n(G/H)$ for some Particular Cubic Extensions

Definition 3.1 — An extension G of H is called normal cubic if H is a normal subgroup of index three.

Let G be a normal cubic extension of H. Let $S = \{e, x, y\} = Z_3$ be an admissible coset representative system. Then there is a map $S \times S \xrightarrow{f} H$ defined as $u.v = f(u, v)\, u \circ v$. Next

$$x.y = f(x, y)\ x\ o\ y = f(x, y)\ e = f(x, y)$$

$$y.x = f(y, x)\ y\ o\ x = f(y, x)\ e = f(y, x)$$

$$x.x = f(x, x)\ x\ o\ x = f(x, x)\ y \text{ i.e. } f(x, x) = x^2y^{-1}$$

$$y.y = f(y, y)\ y\ o\ y = f(y, y)\ x \text{ i.e. } f(y, y) = y^2x^{-1}$$

Further (i) If $f(x, x) = 1$, then $f(x, y) = f(y, x) = f(y, y) = x^3$

(ii) If $f(x, y) = 1$, then $f(x, x) = x^3, f(y, y) = y^3$ and $f(y, x) = 1$.

Thus, we have one of the following cases for an admissible coset representative system S.

(i) $f(u, v) = 1$ for every $u, v \in S = \{e, x, y\}$

(ii) $f(x, x) = 1, f(y, x) = f(x, y) = f(y, y) = x^3 \neq 1$.

(iii) $f(x, y) = f(y, x) = 1, f(x, x) = x^3 \neq 1, f(y, y) = y^3 \neq 1$.

(iv) $f(x, x) \neq 1, f(y, y) \neq 1, f(x, y) \neq 1, f(y, x) \neq 1$.

Remark 3.2 — If G is an abelian group of exponent three, cases (ii) and (iii) will not occur. Hence for such a normal cubic extension G of H, in which G is an abelian group of exponent three, an admissible coset representative system, f, satisfies cases (i) or (iv). We compute $H^n(G/H)$ for such types of extensions. First of all, we require a proposition.

Proposition 3.3 — Let G be a normal cubic extension of G modulo H. Let $S = \{e, x, y\}$ be an admissible coset representative system. Suppose for $x, y \in S$, $f(x, y) \neq 1, f(y, x) \neq 1$. $f(x, x) \neq 1$ and $f(y, y) \neq 1$. Then the number of non-A-sequences of length n is $3^n - 9(2n - 3)$.

PROOF : In this case a n-sequence $(x_1, x_2, ..., x_n, e, e, ...)$, $x_i \in S$, is an A-sequence iff for $i \geqslant 3$, $x_i = e$ except for at most one i. Hence the required number is $3^n - 9(2n - 3)$.

Theorem 3.4 — Let G be a normal cubic extension of H such that G is an abelian group of exponent three. Then for $n > 3$, $H^n(G/H) \simeq \overset{|H|}{\oplus} \underset{Z(Z_3)}{\text{Ext}^n} (Z, H) \overset{p}{\oplus} H$, $h = |H|$ is the order of H and $p = (h^2 - h)(3^n - 9(2n - 3))$.

PROOF : Since G is a cubic extension of H, the cardinality of admissible coset representative systems S is $|H|^2$. By Remark 3.2, for any $S \in$ S, satisfies one of the follwing conditions :

(i) $f(u, v) = 1$ for every $u, u \in S = \{e, x, y\}$

(ii) $f(x, x) \neq 1, f(x, y) \neq 1, f(y, x) \neq 1$ and $f(y, y) \neq 1$.

Condition (i) will be satisfied by an admissible coset representative system of the form $\{e, x, x^{-1}\}$. The cardinality of such admissible coset representative systems, is obviously h. Let (S, H, σ, f) be a C-groupoid (in fact a C-group for S is a group) determined by such an admissibe coset representative system S. The nth parahomology of (S, H, σ, f) is given by

$$H^n(S, H, C(H)) = \underset{Z(Z_3)}{\mathrm{Ext}^n}(Z, C(H)) = \underset{Z(Z_3)}{\mathrm{Ext}^n}(Z, H) \text{ by}$$

Proposition 1.13 — For the rest (h^2-h) admissible coset representative systems,

$$H^n(S, H, C(H)) \simeq \overset{r}{\oplus}\, C(H) \oplus H^n(B, C(H))$$

$$\simeq \overset{r}{\oplus}\, H \oplus H^n(B.\, H),$$

where $r = 3^n-9(2n-3)$. So that by Proposition 1.10 we have

$$H^n(G/H) \cong \overset{h}{\oplus}\, H^n(S, H, H) \overset{h^2-h}{\oplus} H^n(S, H, H)$$

$$\simeq \overset{h}{\oplus} \underset{Z(Z_3)}{\mathrm{Ext}^n}(Z, H) \overset{h^2-h}{\oplus} \left(\overset{r}{\oplus}\, H \oplus H^n(B, H)\right)$$

$$\simeq \overset{h}{\oplus} \underset{Z(Z_3)}{\mathrm{Ext}^n}(Z, H) \overset{p}{\oplus}\, H \overset{h^2-h}{\oplus} H^n(B, H)$$

To complete the proof, we show that $H^n(B, H) \simeq \{0\}$ for every admissible coset representative system, for which f satisfies case (iv).

Here $B = \{B_n, d_n\}$ where $B_n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in S$ and $(x_1, x_2, \ldots, x_n, e, e\ldots)$ is an A-sequence$\}$. Then we have a complex $X \longrightarrow B^{2n} \xrightarrow{\delta} B^{2n+1} \xrightarrow{\delta} B^{2n+2}$ where δ is defined as $\delta(f) = f \circ d$.

Calculation of Ker $\delta : B^{2n-1} \xrightarrow{\delta} B^{2n}$

Since H is normal, its action on S is trivial. Hence, $\delta f(x_1, x_2, \ldots, x_{2n})$

$$= f(x_2, x_3, \ldots, x_{2n}) + \sum_{t=1}^{2n-1} (-t)^t f(x_1, x_2, \ldots, x_{t-1}, x_t \circ x'_{t+1}, x_{t+2}, \ldots, x_{2n})$$

$$+ f(x_1, x_2, \ldots, x_{2n-1})$$

Suppose $f \in$ Ker δ, then $\delta f(x_1, x_2, \ldots, x_{2n}) = 0$. This gives us following cases :

Case (1) : $[x_1 = e, x_2 = u, x_{i_0} = u', x_i = e$ otherwise, $u, u' \in \{e, x, y\}]$.

(i) If $i_0 \neq 2n$ is even, then $f\left(e, u, \ldots, u^{i_0}, \ldots, e\right) = 0$ ...(1)

(ii) If $i_0 = 2n$, then $f(e, u, \ldots, e) = 0$ ...(2)

(iii) If $i_0 \neq 3$ is odd, then $f\left(e, u, \ldots, u^{i_0 - 1}, \ldots, e\right) = 0$ ...(3)

(iv) If $i_0 = 3$, then $f(e, u \circ u', e, \ldots, e) = 0$ ...(4)

Case (2) : [$x_2 = e$, $x_1 = a$, $x_{i_0} = u$, $x_i = e$ otherwise $a \in \{x, y\}$]

(i) If i_0 is odd, then $f\left(e, e, \ldots, u^{i_0 - 1}, \ldots, e\right) = 0$ ...(5)

(ii) If $i_0 \neq 2n$ is even, then $f\left(e, e, \ldots, u^{i_0 - 1}, \ldots, e\right) - f\left(a, e, \ldots, u^{i_0 - 1}, \ldots, e\right)$

$+ f\left(a, e, \ldots, u^{i_0}, \ldots, e\right) = 0$ ...(6)

(iii) If $i_0 = 2n$ then $f(e, e, \ldots, u) - f(a, e, \ldots, u) + f(a, e, \ldots, e) = 0$ ...(7)

Case (3) : [$x_1 = a$, $x_2 = b$, $x_{i_0} = u$, $x_i = e$ otherwise, $a, b \in \{x, y\}$]

(i) If $i_0 \neq 2n$ is even, then $f\left(b, e, \ldots, u^{i_0 - 1}, \ldots, e\right)$

$- f\left(a \circ b, e, \ldots, u^{i_0 - 1}, \ldots, e\right) + f\left(a, b, \ldots, u^{i_0}, \ldots, e\right) = 0$ ...(8)

(ii) If $i_0 = 2n$, then $f(b, e, \ldots, u) - f(a \circ b, e, \ldots, u)$
$+ f(a, b, \ldots, e) = 0$ ...(9)

(iii) If $i_0 \neq 3$ is odd, then $f\left(b, e, \ldots, u^{i_0 - 1}, \ldots, e\right)$

$- f\left(a \circ b, e, \ldots, u^{i_0 - 1}, \ldots, e\right) + f\left(a, b, \ldots, u^{i_0 - 1}, \ldots, e\right) = 0$ ...(10)

(iv) If $i_0 = 3$, then $f(b, u\, e, e, \ldots, e) - f(a \circ b, u', e, \ldots, e)$
$+ f(a, b \circ u', e, \ldots, e) = 0$ ...(11)

Thus $f \in \operatorname{Ker} \delta$ iff one of the conditions represented by the equations (1) to (11) holds. We can give arbitrary values to the following :

(6) $f\left(v, e, \ldots, u^{i_0 - 1}, \ldots, e\right)$ $i_0 \neq 2n$ is even $u \neq e$, $v \in \left\{e, x, y\right\}$

(7) $f(v, e, \ldots\ldots\ldots, u)$, $u \neq e$, $v \in \{e, x, y\}$

and rest are dependent on these. Suppose f is a member of Ker δ such that

$$(6)\ f\left(v, e, \ldots, u^{i_0-1}, \ldots, e\right) = \alpha_1^{i_0-1},\ i_0 \neq 2n \text{ is even, } u \neq e,\ v \in \left\{e, x, y\right\}$$

$$(7)\ f(e, e, \ldots\ldots\ldots\ldots\ldots, u) = \alpha_3\ u \neq e,\ v \in \{e, x, y\}$$

Then take $g \in B^{2n-2}$ defined by

$$g\left(e, e, \ldots, u^{i_0-2}, \ldots, e\right) - g\left(v, e, \ldots, u^{i_0-1}, \ldots, e\right) = \alpha_1^{i_0-1},\ i_i \neq 2n \text{ is even}$$

$$u \neq e,\ v \in \{e, x, y\}$$

$$g(e, e, \ldots\ldots\ldots, u') - g(v, e, \ldots\ldots\ldots, e) = \alpha_3,\ u \neq e,\ v \in \{e, x, y\}$$

Then consider $\left(B^{2n-2} \xrightarrow{\delta} B^{2n-1} \xrightarrow{\delta} B^{2n} \right)$

$$\delta g\left(v, e, \ldots, u^{i_0-1}, \ldots, e\right) = g\left(e, e, \ldots, u^{i_0-2}, \ldots, e\right) - g\left(v, e, \ldots, u^{i_0-1}, \ldots, e\right)$$

$$= \alpha_1^{i_0-1} = f\left(v, e, \ldots, u^{i_0-1}, \ldots, e\right) u \neq e,\ i_0 \neq 2n \text{ is even } v \in \{e, x, y\}$$

$$\delta g(v, e, \ldots, u) = g(e, e, \ldots, u) - g(v, e, \ldots\ldots, e)$$

$$= \alpha_3 = f(v, e, \ldots, u),\ u \neq e,\ v \in \{e, x, y\}$$

From these, it follows that Ker = Image B^{2n-2}. Hence
$\overline{H}^k(B, H) = \{0\}$ if $k > 3$ is odd. Similarly $H^k(B, H) = \{0\}$ if $k > 2$ is even. Hence $H^n(B, C(H)) = \{0\}$ for $n > 3$. Now the result follows :

Corollary 3.6—Let G be a normal cubic extension of H such that G is of exponent three. Then for $n > 3$, $H^n(G/H) \simeq \overset{\lambda}{\oplus} H$ where

$$\lambda = ((h^2 - h)(3^n - 9(2n - 3)) + h).$$

PROOF : The proof follows from the fact that

$$H \simeq \overset{\alpha}{\oplus} Z_3 \text{ and } \underset{Z(Z_3)}{\mathrm{Ext}^n} (Z, Z_3) \simeq Z_3$$

4. Biquadratic Extension of Exponent Two

An extension G of H is called a biquadratic extension of exponent two, if G is a group of exponent two and $[G : H]$ is four. Let G be such an extension of H and S be the set of admissible coset representative systems. Then each $S = \{e, x, y, z\} \simeq Z_2 \oplus Z_2$. There is a map $S \times S \xrightarrow{f} H$ defined by $u, v = f(u, v)\ u\ o\ v$. Then

(i) $f(x, x) = f(y, y) = f(z, z) = 1$

(ii) $f(u, v) = x\,y\,z$ if $u, v \in \{x, y, z\}$ and $u \neq v$.

Thus for $S \in \mathbf{S}$, f satisfies one of the following conditions.

(1) $f(u, v) = 1$, for every $u, v \in S$.

(2) $f(x, x) = f(y, y) = f(z, z) = 1$ and
$f(x, y) = f(y, x) = f(x, z) = f(z, x) = f(y, z) = f(z, y) = xyz \neq 1$.

For a C-group (S, H, σ, f) for which f satisfies the case (1), we have by Proposition 1.13.

$$H^n(S, H, C(H)) = H^n(S, H, H) \simeq \operatorname{Ext}^n_{Z(V_4)}(Z, H) \simeq H \qquad \text{...(A)}$$

Proposition 4.1—Let G be a biquadratic extension of exponent two and $S = \{e, x, y, z\}$ be an admissible coset representative system such that $f(u, v) \neq 1$ if $u \neq e$, $v \neq e$, $u \neq v$; $u, v \in S$. Then the cardinality of non A-sequences of length n is $4^n - 16\,(1 + 3(2^{n-2} - 1))$.

PROOF : A-n-sequence $(x_1, x_2, \ldots, x_n, e, e, \ldots)$ is an A-sequence iff there exists at most $i_1, i_2, \ldots, i_r$, $i_r \geqslant 3$ for every r such that

$x_{i_1} = x_{i_2} = \ldots = x_{i_r} = x$ or y or z and $x_j = e$ otherwise.

The required number, is, therefore, $4^n - 16(1 + 3(2^{n-2} - 1))$.

Let (S, H, σ, f) be a C-group for which f satisfies (2). By Propositions 1.10 and 4.1 we have

$$H^n(S, H, C(H)) = H^n(S, H, H) \simeq \overset{p}{\oplus}\, H \oplus H^n(B, H) \text{ where}$$

$$p = 4^n - 16(1 + 3(2^{n-2} - 1)).$$

Proposition 4.2 — Let G be a biquadratic extension of H of exponent two. Then there are $h^2 = |\,H\,|^2$ admissible coset representative systems for which $f(u, v) = 1$, always $u, v \in S = \{e, x, y, z\}$

Theorem 4.3 — Let G be a biquadratic extension of H of exponent two. Then $\overset{m}{\oplus}\, H$, $m = h^2 + (h^3 - h^2)\,(4^n - 16(1 + 3(2^{n-2} - 1))$ is direct summand of $H^n\,(G/H)$.

PROOF : $H^n(G/H) \simeq \overset{h^2}{\oplus}\, H^n(S, H, H) \overset{h^3-h^2}{\oplus} H^n(S, H, H)$.

For the h^2 C-group $H^n(S, H, H)$ is given by (A) and for the rest $(h^3 - h^2)$ C-groups $H^2(S, H, H)$ is given by (B). The result now follows.

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