STABILITY OF A LIQUID LAYER FLOWING DOWN A PERMEABLE BOUNDARY

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An analysis is presented for the stability of a layer of a viscous liquid flowing down a permeable boundary taking account of velocity slip at the surface of the porous bounding wall. The slip at the wall is found to be destabilizing.

1. INTRODUCTION

The stability characteristics of a layer of a viscous liquid flowing down an inclined plane have been theoretically investigated by Yih (1955, 1963) and Benjamin (1957) and experimentally by Jackson (1955) and Binnie (1957). The flow is found to be unstable for disturbances of wave lengths large compared with the thickness of the layer at relatively small Reynolds numbers.

The purpose of the present investigation is to study the above problem when the bounding wall on which the liquid is flowing is adjacent to a porous medium. It has been found experimentally by Beavers and Joseph (1967) and Beavers *et al.* (1970) that when water (or oil) flows in a parallel plate channel one of whose walls is a porous medium, there is a velocity slip at the porous wall proportional to the wall velocity gradient. In fact Beavers and Joseph (1967) have shown that the shear effects are transmitted into the permeable medium through a boundary layer region, while Beavers *et al.* (1970) have demonstrated that, in some cases, the slip velocity may become as large as 60% of the mean velocity in the channel. In the present analysis, we shall, therefore, allow for slip velocity at the porous boundary. The medium adjacent to this boundary is assumed to be homogeneous and isotropic so that the flow in this medium is governed by Darcy's law. Disturbances are assumed to be two-dimensional and the perturbation technique of Yih (1963) will be followed in the stability analysis.

2. MATHEMATICAL FORMULATION AND THE STABILITY ANALYSIS

A layer of an incompressible viscous fluid of thickness h flows down a plane inclined at an angle β to the horizon (Fig. 1). In a rectangular co-ordinate system (x, y, z) the basic flow is parallel to x-axis and y-axis is normal to the plane directed

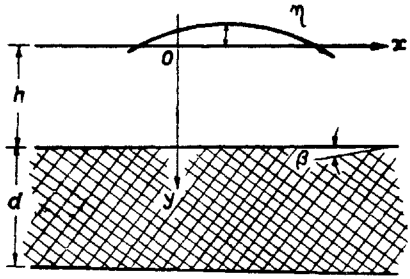

FIG. 1. A sketch of the physical problem.

downwards, the origin being taken on the undisturbed free surface. The basic velocity $\bar{u}(y)$ satisfies

$$\nu \frac{d^2\bar{u}}{dy^2} + g \sin \beta = 0 \quad \text{...(2.1)}$$

subject to vanishing tangential stress condition at the free surface

$$\frac{d\bar{u}}{dy} = 0 \qquad \text{at } y = 0. \quad \text{...(2.2)}$$

As for the condition at the permeable boundary, we follow the model of Beavers and Joseph (1967) as

$$-\frac{d\bar{u}}{dy} = \frac{\alpha_1}{k^{1/2}} (\bar{u} - Q) \qquad \text{at } y = h \quad \text{...(2.3)}$$

where α_1 is a dimensionless constant depending on the porous material, and k is the permeability and Q represents the filter velocity in the porous medium. A typical example of such a medium is a porous material consisting of lattice work of metallic fibres. In the presence of gravity, Q is given by Darcy's law as

$$Q = \frac{k}{r} \rho g \sin \beta. \quad \text{...(2.4)}$$

The solution of eqn. (2.1) satisfying (2.2) and (2.3) is

$$\frac{\bar{u}(y)}{U_m} = \frac{1}{2} \left(1 - \frac{y^2}{h^2} \right) + M \quad \text{...(2.5)}$$

where

$$U_m = \frac{gh^2 \sin \beta}{\nu}, \; M = \frac{k^{1/2}}{\alpha_1 h} + \frac{k}{\rho h^2}. \quad \text{...(2.6)}$$

It can be seen from (2.5) that the slip velocity at the permeable wall given by $\bar{u}(h)$ increases with increase in M.

To study the stability of the basic flow given by (2.5), we assume two-dimensional disturbances to this flow and take the perturbed velocity components as $(\bar{u}(y) + u', v')$ and the perturbed pressure as $\bar{p} + p'$, $\bar{p}$ being the basic pressure distribution. Substituting these in the equations of momentum and continuity and linearizing, we get

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu \nabla^2 u' \qquad ...(2.7)$$

$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu \nabla^2 v' \qquad ...(2.8)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial x} = 0. \qquad ...(2.9)$$

We introduce the Reynolds number $R = U_m h/\nu$ and write the foregoing equations in dimensionless form by using the following dimensionless variables

$$u = \frac{\nu u'}{h^2 g \sin\beta}, \; v = \frac{\nu v'}{h^2 g \sin\beta}, \; \tau = \frac{tgh \sin\beta}{\nu}$$

$$X = \frac{x}{h}, \; Y = \frac{y}{h}. \qquad ...(2.10)$$

Equation (2.9) then permits the use of a dimensionless stream function $\psi(X, Y, \tau)$ defined by

$$u = \frac{\partial \psi}{\partial Y}, \; v = -\frac{\partial \psi}{\partial X}. \qquad ...(2.11)$$

Assuming

$$\psi = \varphi(Y) \exp[i\alpha(X - C\tau)] \qquad ...(2.12)$$

and eliminating pressure between eqns. (2.7) and (2.8), we obtain the Orr-Sommerfeld equation in φ as

$$\varphi^{iv} - 2\alpha^2 \varphi'' + \alpha^4 \varphi = i\alpha R\,[\{U(Y) - c\}(\varphi'' - \alpha^2 \varphi) - U''\varphi] \qquad ...(2.13)$$

where a prime denotes differentiation with respect to Y, and $U(Y) = \bar{u}(y)/U_m$.

Let η be the displacement of the free surface from its mean position, so that the kinematic condition at the free surface is

$$\frac{\partial \eta}{\partial t} + \bar{u}(0)\frac{\partial \eta}{\partial x} = v'(0) \qquad ...(2.14)$$

Assuming $\xi = \eta/h$ proportional to $\exp[i\alpha(X - C\tau)]$ and using eqns. (2.5), (2.10), (2.11) and (2.12), the above equation leads to

$$\xi = \frac{\varphi(0)}{c'} \exp\,[i\alpha(X - C\tau)],\; C' = C - \frac{1}{2} - M. \qquad \ldots(2.15)$$

Now the shear stress at the free surface due to the disturbance is

$$\tau_{xy} = r\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) - \rho g h \xi \sin\beta \qquad \ldots(2.16)$$

where the last term gives the effect of the variation of mean shear due to the deviation of the free surface from its mean position. Since $\tau_{xy} = 0$ at the free surface, eqn. (2.16) yields along with eqns. (2.10), (2.11), (2.12) and (2.15),

$$\varphi''(0) + \left(\alpha^2 - \frac{1}{C'}\right)\varphi(0) = 0. \qquad \ldots(2.17)$$

Again, at the free surface the normal stress balances that due to surface tension T. This gives

$$-\eta\frac{dp}{dy} - p' + 2r\frac{\partial v'}{\partial y} + T\frac{\partial^2 \eta}{\partial x^2} = 0 \qquad \ldots(2.18)$$

at the free surface. Elimination of p' between (2.7) and (2.18) gives upon using (2.10), (2.11), (2.12) and (2.15)

$$\varphi'''(0) + (i\alpha C'R - 3\alpha^2)\varphi'(0) + (i\alpha \cot\beta + i\alpha^3 SR)\frac{\varphi(0)}{C'} = 0, \qquad \ldots(2.19)$$

where $S = T\nu^2/\rho h^5 g^2 \sin^2\beta$.

Boundary conditions at the porous wall are rather complicated due to the coupling of the flow inside the porous medium with the flow outside it. In the disturbed state we take the perturbation velocity components and pressure as (u_p', v_p') and p_p' (subscript p referring to the porous medium). Then by Darcy's law

$$u_p' = -\frac{k}{r}\frac{\partial p_p'}{\partial x},\; v_p' = -\frac{k}{r}\frac{\partial p_p'}{\partial y} \qquad \ldots(2.20)$$

which imply by virtue of the equation of continuity

$$\frac{\partial^2 p_p'}{\partial x^2} + \frac{\partial^2 p_p'}{\partial y^2} = 0. \qquad \ldots(2.21)$$

Introducing $P_p = p_p'/\rho U_m^2$ and assuming

$P_p = f(Y)\, e^{i\alpha\,(X - C\tau)}$, equation (2.21) gives

$$f(Y) = A_1\, e^{\alpha Y} + B_1\, e^{-\alpha Y}. \qquad \ldots(2.22)$$

Since the porous medium has an impermeable underside at $y = d + h$ where v_p' vanishes, we have from (2.20) and (2.22)

$$B_1 = A_1\, e^{2\alpha}\left(1 + \frac{d}{h}\right) \qquad \text{...(2.23)}$$

Hence from (2.20)

$$v_p' = -\frac{\alpha k \rho U_m^2}{rh} A_1 \left[e^{\alpha Y} - e^{2\alpha(1+(d/h))-\alpha Y}\right] e^{i\alpha(X-Ct)}. \qquad \text{...(2.24)}$$

In the viscous flow region outside the porous medium

$$v' = -U_m\, \varphi(Y)\, i\alpha\, e^{i\alpha(X-Ct)}. \qquad \text{...(2.25)}$$

By continuity of flow at $y = h$, we must have

$$v_p'(h) = v'(h), \qquad \text{...(2.26)}$$

which gives from (2.24) and (2.25)

$$A_1 = \frac{irh\, \varphi(1)}{k\rho\, U_m\, e^{\alpha}[1 - e^{2\alpha d/h}]}. \qquad \text{...(2.27)}$$

In the perturbed state, equation (2.3) becomes

$$-\frac{\partial u'}{\partial y} = \frac{\alpha_1}{k^{1/2}} (u' - u_p') \text{ at } y = h. \qquad \text{...(2.28)}$$

Invoking (2.10), (2.11), (2.12) and (2.20) together with the expression for p_p' deduced above, equation (2.28) gives

$$\varphi'(1) + \frac{\alpha(e^{2\alpha d/h} + 1)}{(e^{2\alpha d/h} - 1)} \cdot \varphi(1) + N\varphi''(1) = 0, \qquad \text{...(2.29)}$$

where

$$N = \sqrt{k}/\alpha_1 h. \qquad \text{...(2 30)}$$

Finally the normal stress must be continuous at the porous boundary $y = h$. This demands

$$-p_p' + 2r\frac{\partial v_p'}{\partial y} = -p' + 2r\frac{\partial v'}{\partial y} \quad \text{at} \quad y = h\,. \qquad \text{...(2.31)}$$

It may be noted that in view of the slip at $y = h$, $\partial u'/\partial x$ and hence $\partial v'/\partial y$ cannot vanish there. Eliminating p' between (2.7) and (2.31) and using (2.10), (2.11), (2.12) and (2.20) we get after a lengthy algebra

$$\left[i\alpha R + \frac{\alpha(e^{2\alpha d/h} + 1)}{(e^{2\alpha d/h} - 1)\,\alpha_1^2 N^2} + \frac{2\alpha^3(e^{2\alpha d/h} + 1)}{(e^{2\alpha d/h} - 1)}\right]\varphi(1)$$

$$+[i\alpha R(N + \alpha_1^2 N^2) - i\alpha CR + 3\alpha^2]\,\varphi'(1) - \varphi'''(1) = 0. \qquad \text{...(2.32)}$$

It may be noticed from (2.32) that to $O(\alpha)$, the terms $2r\,\partial v_p'/\partial y$ and $2r\,\partial v'/\partial y$ will not contribute to this boundary condition and we have effectively $p_p' = p'$ at $y = h$ as is usually assumed in the literature.

Equation (2.13) subject to the boundary condition (2.17), (2.19), (2.29) and (2.32) constitutes an eigenvalue problem. For a non-trivial solution a relation

$$C = C(R, \alpha, \alpha_1, N, d/h) \qquad \ldots(2.33)$$

must hold from which the curves of neutral stability $C_i = 0$ (where $C = C_r + i\,C_i$) can be obtained.

3. Solution for Long Waves

Since the order of the differential equation (2.13) is not lowered and the boundary conditions do not degenerate as $\alpha \to 0$ we may solve this differential system by a regular perturbation method taking α as a small parameter. In the first approximation we put $\alpha = 0$ in (2.13), (2.17), (2.19), (2.29) and (2.32), so that the eigenfunction $\varphi_0(Y)$ satisfies

$$\varphi_0^{iv}(Y) = 0 \qquad \ldots(3.1)$$

subject to

$$\varphi_0''(0) = \frac{\varphi_0(0)}{C_0'}, \quad \varphi_0'''(0) = 0$$

$$\varphi_0'(1) + \frac{h}{d}\,\varphi_0(1) + N\,\varphi_0''(1) = 0$$

$$\frac{1}{\alpha_1^2 N^2}\cdot\frac{h}{d}\,\varphi_0(1) - \varphi_0'''(1) = 0. \qquad \ldots(3.2)$$

In deriving the last two boundary conditions in (3.2) from (2.29) and (2.32), use is made of the relation

$$\frac{\alpha\,(e^{2\alpha d/h} + 1)}{(e^{2\alpha d/h} - 1)} = \frac{h}{d}\,[1 + O(\alpha^2)]. \qquad \ldots(3.3)$$

Here C_0' is connected with C_0, the zeroeth eigenvalue through (2.15) as

$$C_0' = C_0 - \frac{1}{2} - M = C_0 - \frac{1}{2} - (N + \alpha_1^2\,N^2). \qquad \ldots(3.4)$$

The solution of the above system gives

$$\varphi_0(Y) = 1 - \frac{2(1 + N)}{1 + 2N}\,Y + \frac{1}{1 + 2N}\,Y^2\,;\; C_0' = \frac{1}{2}\,(1 + 2N)\,, \qquad \ldots(3.5)$$

where the multiplicative constant in $\varphi_0(Y)$ is assumed unity in view of the linearity and the homogeneity of the differential system. Since from (3.4) and (3.5) $C_0 = 1 + N + M$ is real, it follows that $\alpha = 0$ is a part of the neutral stability curve.

To see how the eigenvalue is modified as the wave number increases from zero, we consider the second approximation $\varphi_1(Y)$ by retaining terms of order α. Hence correct to $O(\alpha)$, equation (2.13) gives on using (2.5), (2.6) and (3.5)

$$\varphi_1^{iv}(Y) = -\frac{2i\alpha\, R(1+N)}{(1+2N)} Y. \qquad \text{...(3.6)}$$

The corresponding boundary conditions can be written from (2.17), (2.19), (2.29) and (2.32) as

$$\varphi_1''(0) - \frac{1}{C_0'}\varphi_1(0) + \frac{\triangle C}{C_0'^2}\cdot \varphi_0(0) = 0 \qquad \text{...(3.7)}$$

$$\varphi_1'''(0) + i\alpha\left[R\, C_0'\, \varphi_0'(0) + \cot\beta\, \frac{\varphi_0(0)}{C_0'}\right] = 0 \qquad \text{...(3.8)}$$

$$\varphi_1'(1) + \frac{h}{d}\varphi_1(1) + N\varphi_1''(1) = 0 \qquad \text{...(3.9)}$$

$$i\alpha R\varphi_0(1) + \frac{h}{d\alpha_1^2 N^2}\varphi_1(1) + [i\alpha\, R(N + \alpha_1^2 N^2) - i\alpha\, RC_0]\,\varphi_0'(1) - \varphi_1'''(1) = 0 \qquad \text{...(3.10)}$$

where $\triangle C$ stands for the change in C_0 as α increases from zero.

Integrating (3.6) successively four times, we get

$$\varphi_1(Y) = A_2\frac{Y^3}{6} + A_3\frac{Y^2}{2} + A_4 Y + A_5 - \frac{i\alpha R(1+N)}{60\,(1+2N)} Y^5 \qquad \text{...(3.11)}$$

where A_2, A_3, A_4 and A_5 are constants. The boundary conditions (3.7),—(3.10) lead to

$$A_5 = \frac{(1+2N)\, A_3}{2} + \frac{2\triangle C}{(1+2N)} \qquad \text{...(3.12)}$$

$$A_2 = i\alpha\left[R(1+N) - \frac{2\cot\beta}{(1+2N)}\right] \qquad \text{...(3.13)}$$

$$\frac{A_2}{2} + A_3 + A_4 - \frac{i\alpha R(1+N)}{12\,(1+2N)} + \frac{h}{d}\left[\frac{A_2}{6} + \frac{A_3}{2} + A_4 + A_5 - \frac{i\alpha R(1+N)}{60\,(1+2N)}\right] + \left[A_2 + A_3 - \frac{i\alpha R(1+N)}{3\,(1+2N)}\right] N = 0 \qquad \text{...(3.14)}$$

$$\frac{h}{d\alpha_1^2 N^2}\left[\frac{A_2}{6} + \frac{A_3}{2} + A_4 + A_5 - \frac{i\alpha\, R(1+N)}{60\,(1+2N)}\right] = A_2 - i\alpha R(1+N) \qquad \text{...(3.15)}$$

where use is made of (3.5).

Substituting (3.15) in (3.14) and using (3.13) we obtain

$$(1+N)\,A_3 + A_4 = -\left(\frac{1}{2} + N + \alpha_1^2 N^2\right) i\alpha\left[R(1+N) - \frac{2\cot\beta}{(1+2N)}\right] + i\alpha R(1+N)\left[\frac{1+4N}{12\,(1+2N)} + \alpha_1^2 N^2\right] \quad ...(3.16)$$

and from (3.12), (3.13) and (3.15) we have

$$(1+N)\,A_3 + A_4 = i\alpha\left[R(1+N) - \frac{2\cot\beta}{(1+2N)}\right]\left[\frac{\alpha_1^2 N^2 d}{h} - \frac{1}{6}\right] + i\alpha R\,(1+N)\left[\frac{1}{60\,(1+2N)} - \frac{\alpha_1^2 N^2 d}{h}\right] - \frac{2\triangle C}{(1+2N)}\,. \quad ...(3.17)$$

Equating (3.16) and (3.17), the following relation is obtained

$$-\left(\frac{1}{2} + N + \alpha_1^2 N^2\right).\, i\alpha\left[R(1+N) - \frac{2\cot\beta}{1+2N}\right] + i\alpha R(1+N)\left[\frac{1+4N}{12\,(1+2N)} + \alpha_1^2 N^2\right]$$

$$= i\alpha\left[R(1+N) - \frac{2\cot\beta}{(1+2N)}\right]\left[\frac{\alpha_1^2 N^2 d}{h} - \frac{1}{6}\right] + i\alpha R(1+N)\left[\frac{1}{60\,(1+2N)} - \frac{\alpha_1^2 N^2 d}{h}\right] - \left[\frac{2\triangle C}{(1+2N)}\right]. \quad ...(3.18)$$

This shows that while $C_i = 0$ at $\alpha = 0$, C_i will increase or decrease as α increases from zero according as $\triangle C \gtrless 0$. Thus the critical Reynolds number R_C at the onset of instability is obtained by putting $\triangle C = 0$ in (3.18). This gives

$$R_C = \frac{5\left[1 + 3N + 3\alpha_1^2 N^2\left(1 + \frac{d}{h}\right)\right]\cot\beta}{(1+N)(2 + 10\,N + 15\,N^2)}. \quad ...(3.19)$$

This shows that for fixed α_1, N and β, R_C increases with increase in d/h which means that for a given thickness h of the liquid layer increase in the thickness of the porous medium has a stabilizing influence on the flow. We have plotted R_C against $\sqrt{k}/h$ [remembering that $N = \sqrt{k}/\alpha_1 h$ as defined in (2.30)] from (3.19) for several values of α_1 with $\beta = \pi/4$ and $d/h = 1$, and the results are shown in figure 2, values of $\sqrt{k}/h$ are taken small in the computation since the assumption $h \gg k^{1/2}$ is needed to ensure that the basic flow is unidirectional as given by (2 5). If on the other hand $\sqrt{k}$ is of order h or more, then the average size of the individual pores in the porous medium is at least of the order of the thickness h of the fluid layer above it and then the assumption of a basic unidirectional flow breaks down. It can be seen from the figure that for fixed α_1, R_C decreases with increase in $\sqrt{k}/h$ so that increase in permeability has a destabilizing effect on the flow. Further for fixed permeability, R_C increases

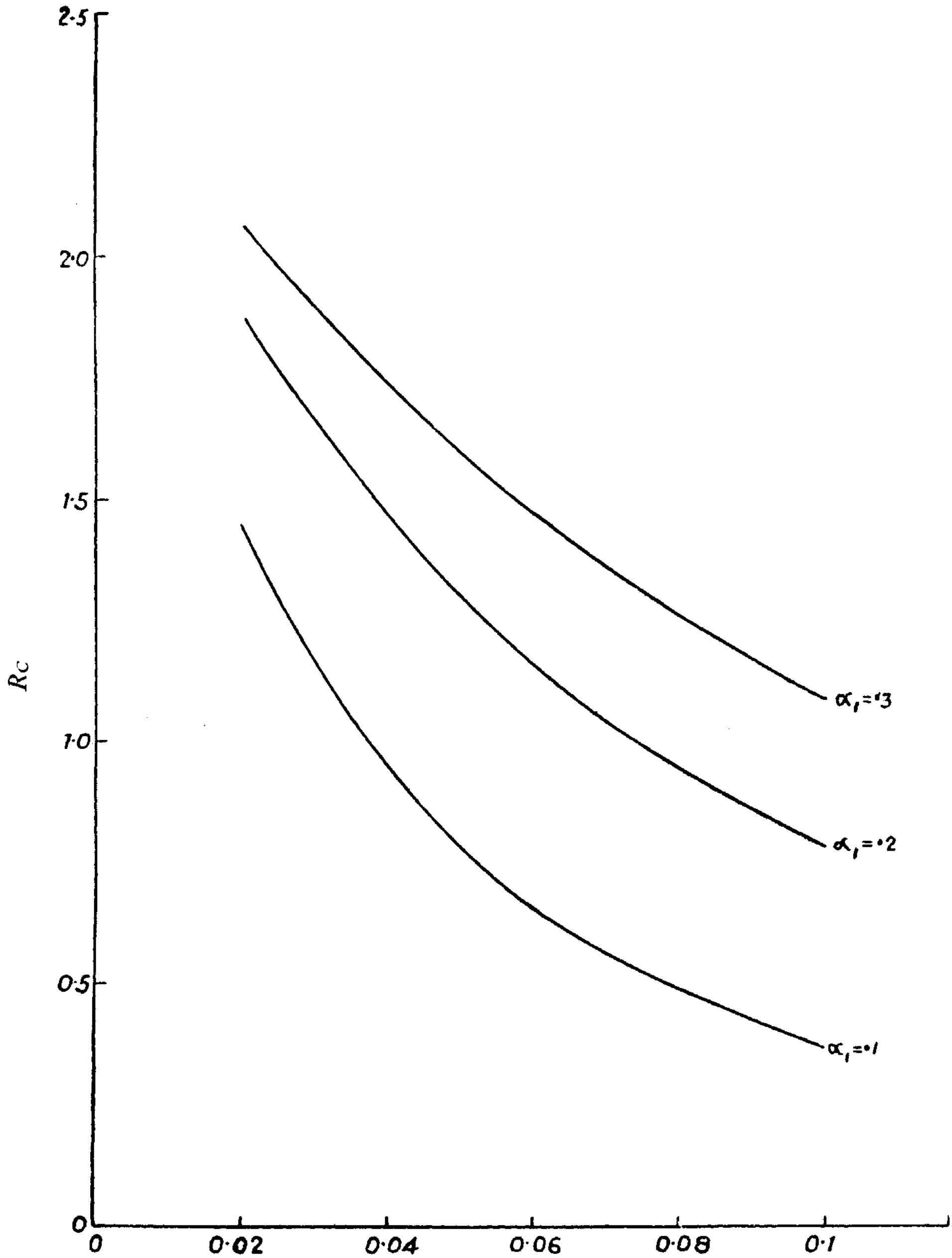

FIG. 2. Variation of R_C with $\sqrt{k}/h$ for different values of α_1.

with increase in α_1. The values of α_1 are taken to be order 1/10 e.g. for a porous material consisting of a lattice work of metallic fibers and having a permeability $k = 5.1 \times 10^{-7}$ cm^2, $\alpha_1 = 0.1$ (Sparrow *et al.* 1971).

4. Discussion

The presence of slip at the porous boundary causes a decrease in the wall shear and this may perhaps tend to destabilize the flow through the diminution of the retaining influence of the wall. The stabilizing role of α_1 is, however, not well understood. Beavers and Joseph experimentally found that $\alpha_1/k^{1/2}$ is practically independent of viscosity. While the permeability k depends on the size and shape of the interstices, α_1 is a dimensionless quantity depending on the material in the porous medium. Fig. 2 shows that for small enough values of $\sqrt{k}/h$, R_C is of order 2 so that the critical Reynolds number based on a length representative of the size of the pores will indeed be very small. This is in keeping with the fact that in the case of percolation through a porous medium the inertia forces are negligible compared with the viscous forces. This requirement is generally met in soil water movement.

Finally, we point out two limitations in our stability analysis. First our investigation is confined to two-dimensional disturbances in anticipation of Squire's theorem. Secondly, the present stability analysis is confined to that of surface waves only.

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