ON THE STUDY OF RAYLEIGH-TYPE WAVES IN A LAYER OF FINITE THICKNESS SANDWICHED BETWEEN TWO SEMI-INFINITE MEDIA

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Propagation of Rayleigh-type waves in a layer sandwiched between two semi-infinite media of compressible material has been studied in this paper. Three different models have been considered taking the inner layer to be composed of (i) compressible material, (ii) incompressible material and (iii) material having cubic symmetry. Wave-velocity equations are obtained in each case and some exact solutions are found. In the first case, the dispersive nature of Rayleigh-type waves is compared with the one obtained by Sezawa and Nishimura (1928).

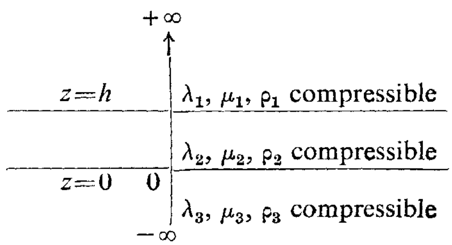
INTRODUCTION

Sezawa and Nishimura (1928) studied the propagation of Rayleigh-type waves along an isotropic homogeneous stratum sandwiched between two semi-infinite homogeneous media, the elastic constants for both the media being the same. They obtained the period equation as a fourth order determinantal equation and studied both symmetric and asymmetric waves with respect to the central plane of the stratum. They have shown that Rayleigh-type waves propagated along an inner stratum are generally dispersive in nature. Lamb (1916) investigated a problem on waves in an elastic plate in which he showed that waves of various lengths are transmitted over a plate. Dispersion of surface waves in multilayered media was studied by Haskell (1953). Harkrider (1964) also investigated the problem of surface waves in multilayered media. Stoneley (1924) investigated the problem of propagation of generalised type of Love waves considering a model composed of a homogeneous layer of finite thickness sandwiched between two semi-infinite isotropic media. Singh (1965) studied the existence of Rayleigh waves in an axially symmetric heterogeneous layer lying between two semi-infinite isotropic media. He obtained the frequency equation in terms of Whittaker functions and did not give any numerical result. In a recent paper, Sinha (1972) investigated the existence of Rayleigh waves in a heterogeneous layer lying between two semi-infinite media. In this paper, we have considered three models of similar type taking the inner stratum to be composed of (i) compressible material, (ii) incompressible material and (iii) material having cubic symmetry. The propagation of elastic waves in material having cubic symmetry has been studied by a few authors (Hearmon 1961). The material in each of the two semi-infinite media is taken to be compressible in all the three cases. Frequency equation has been obtained in each case and numerical results given. The results obtained in case (i) have been compared with those of Sezawa and Nishimura.

Solution of the Problem

Case I

Let us consider an infinitely extended layer of compressible material and of uniform thickness h sandwiched between two semi-infinite compressible media. The origin is chosen at the lower interface and the z-axis perpendicular to either interface so that the upper interface has for its equation $z = h$.



Model considered in Case I

The equations of motion for compressible material in two dimensions are

$$(\lambda_i + \mu_i)\frac{\partial \Delta_i}{\partial x} + \mu_i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)u_i = \rho_i\frac{\partial^2 u_i}{\partial t^2}, \qquad \ldots(1)$$

$$(\lambda_i + \mu_i)\frac{\partial \Delta_i}{\partial z} + \mu_i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)w_i = \rho_i\frac{\partial^2 w_i}{\partial t^2}, \qquad \ldots(2)$$

where u_i and w_i are the components of displacement in the x and z directions, ρ_i is the density and

$$\Delta_i = \frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z}, \ (i = 1, 2, 3);$$

$i = 1$ and 3 correspond to the upper and lower semi-infinite media whereas $i = 2$ corresponds to the layer.

$$\left.\begin{aligned} \text{With} \quad u_i &= \frac{\partial \varphi_i}{\partial x} + \frac{\partial \psi_i}{\partial z} \\ \text{and} \quad w_i &= \frac{\partial \varphi_i}{\partial z} - \frac{\partial \psi_i}{\partial x} \end{aligned}\right\} \qquad \ldots(3)$$

$$\left.\begin{aligned} \text{where} \quad \varphi_i &= A\,\Phi_i(z)\cos k(x-ct) \\ \text{and} \quad \psi_i &= B\,\Psi_i(z)\sin k(x-ct) \end{aligned}\right\} \qquad \dots(4)$$

we obtain

$$\left.\begin{aligned} \varphi_1 &= A_1\,e^{-kn_1 z}\cos k(x-ct) \\ \psi_1 &= B_1\,e^{-ks_1 z}\sin k(x-ct) \end{aligned}\right\} \qquad \dots(5)$$

$$\left.\begin{aligned} \varphi_2 &= (A_2\,e^{-kn_2 z} + A_2'\,e^{kn_2 z})\cos k(x-ct) \\ \psi_2 &= (B_2\,e^{-ks_2 z} + B_2'\,e^{ks_2 z})\sin k(x-ct) \end{aligned}\right\} \qquad \dots(6)$$

$$\left.\begin{aligned} \varphi_3 &= A_3\,e^{kn_3 z}\cos k(x-ct) \\ \psi_3 &= B_3\,e^{ks_3 z}\sin k(x-ct) \end{aligned}\right\} \qquad \dots(7)$$

$$\left.\begin{aligned} \text{where} \quad n_i &= \left(1-\frac{c^2}{\alpha_i^2}\right)^{1/2} \\ \text{and} \quad s_i &= \left(1-\frac{c^2}{\beta_i^2}\right)^{1/2} \end{aligned}\right\} \qquad \dots(8)$$

$$(i = 1, 2, 3).$$

The stress-strain relations are

$$\left.\begin{aligned} \left(\widehat{zz}\right)_i &= \lambda_i\frac{\partial^2\varphi_i}{\partial x^2} + (\lambda_i + 2\mu_i)\frac{\partial^2\varphi_i}{\partial z^2} \\ &\qquad - 2\mu_i\frac{\partial^2\psi_i}{\partial x\partial z} \\ \text{and} \quad \left(\widehat{xz}\right)_i &= \mu_i\left(2\frac{\partial^2\varphi_i}{\partial z\partial x} + \frac{\partial^2\psi_i}{\partial z^2} - \frac{\partial^2\psi_i}{\partial x^2}\right), \end{aligned}\right\} \qquad \dots(9)$$

$$(i = 1, 2, 3).$$

The boundary conditions are

$$\left.\begin{aligned} u_1 &= u_2\,, \\ w_1 &= w_2\,, \\ \left(\widehat{zz}\right)_1 &= \left(\widehat{zz}\right)_2, \\ \left(\widehat{xz}\right)_1 &= \left(\widehat{xz}\right)_2, \end{aligned}\right\} \text{ at } z = h \qquad \dots(10a)$$

and

$$\left.\begin{aligned} u_2 &= u_3 ,\\ w_2 &= w_3 ,\\ \left(\widehat{zz}\right)_2 &= \left(\widehat{zz}\right)_3 ,\\ \left(\widehat{zx}\right)_2 &= \left(\widehat{zx}\right)_3 , \end{aligned}\right\} \text{ at } z = 0. \qquad \ldots(10b)$$

Equations (3), (5), (6), (7) and (9) with the boundary conditions (10 *a*, *b*) give

$$A_1\, e^{-kn_1h} + B_1 s_1\, e^{-ks_1h} - A_2\, e^{-kn_2h} - A_2'\, e^{kn_2h} - B_2 s_2\, e^{-ks_2h} + B_2' s_2\, e^{ks_2h} = 0 \qquad \ldots(11)$$

$$A_1 n_1\, e^{-kn_1h} + B_1\, e^{-ks_1h} - A_2 n_2\, e^{-kn_2h} + A_2' n_2\, e^{kn_2h} - B_2\, e^{-ks_2h} - B_2'\, e^{ks_2h} = 0 \qquad \ldots(12)$$

$$A_1 l_1\, e^{-kn_1h} + B_1\, 2\mu_1 s_1\, e^{-ks_1h} - A_2 l_2\, e^{-kn_2h} - A_2' l_2\, e^{kn_2h} - B_2\, 2\mu_2 s_2\, e^{-ks_2h} + B_2'\, 2\mu_2 s_2\, e^{ks_2h} = 0 \qquad \ldots(13)$$

$$A_1 \frac{2\mu_1}{\mu_2} n_1\, e^{-kn_1h} + B_1 \frac{\mu_1}{\mu_2}\left(1 + s_1^2\right) e^{-ks_1h} - A_2\, 2n_2\, e^{-kn_2h} + A_2'\, 2n_2\, e^{kn_2h} - B_2\left(1 + s_2^2\right) e^{-ks_2h} - B_2'\left(1 + s_2^2\right) e^{ks_2h} = 0 \qquad \ldots(14)$$

$$-A_2 - A_2' - B_2 s_2 + B_2' s_2 + A_3 - B_3 s_3 = 0 \qquad \ldots(15)$$

$$-A_2 n_2 + A_2' n_2 - B_2 - B_2' - A_3 n_3 + B_3 = 0 \qquad \ldots(16)$$

$$A_2 l_2 + A_2' l_2 + B_2\, 2\mu_2 s_2 - B_2'\, 2\mu_2 s_2 - A_3 l_3 + B_3\, 2\mu_3 s_3 = 0 \qquad \ldots(17)$$

$$A_2\, 2n_2 - A_2'\, 2n_2 + B_2\left(1 + s_2^2\right) + B_2'\left(1 + s_2^2\right) + A_3 \frac{2\mu_3}{\mu_2} n_3 - B_3\left(1 + s_3^2\right)\frac{\mu_3}{\mu_2} = 0 \qquad \ldots(18)$$

where

$$l_i = (\lambda_i + 2\mu_i)\, n_i^2 - \lambda_i,\ (i = 1, 2, 3). \qquad \ldots(19)$$

From equations (11), (12), (13), (14), (15), (16), (17) and (18) we get the frequency equation (20) :

$$\begin{vmatrix}
1 & s_1 & -e^{-kn_2h} & -e^{kn_2h} & -s_2\,e^{-ks_2h} & s_2\,e^{ks_2h} & 0 & 0 \\
n_1 & 1 & -n_2\,e^{-kn_2h} & n_2\,e^{kn_2h} & -e^{-ks_2h} & -e^{ks_2h} & 0 & 0 \\
l_1 & 2\mu_1 s_1 & -l_2\,e^{-kn_2h} & -l_2\,e^{kn_2h} & -2\mu_2 s_2\,e^{-ks_2h} & 2\mu_2 s_2\,e^{ks_2h} & 0 & 0 \\
2\frac{\mu_1}{\mu_2}n_1 & \frac{\mu_1}{\mu_2}(1+s_1^2) & -2n_2\,e^{-kn_2h} & 2n_2\,e^{kn_2h} & -(1+s_2^2)e^{-ks_2h} & -(1+s_2^2)e^{ks_2h} & 0 & 0 \\
0 & 0 & -1 & -1 & -s_2 & s_2 & 1 & -s_3 \\
0 & 0 & -n_2 & n_2 & -1 & -1 & -n_3 & 1 \\
0 & 0 & l_2 & l_2 & 2\mu_2 s_2 & -2\mu_2 s_2 & -l_3 & 2\mu_3 s_3 \\
0 & 0 & 2n_2 & -2n_2 & 1+s_2^2 & 1+s_2^2 & \frac{2\mu_3 n_3}{\mu_2} & -\frac{\mu_3}{\mu_2}(1+s_3^2)
\end{vmatrix} = 0 \qquad \ldots(20)$$

Case II

If we replace the inner layer of compressible material considered in Case I by a layer of the same thickness but of incompressible material, then

$$\triangle_2 \equiv \frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0. \qquad ...(21)$$

Assuming that $-\pi = \lim \lambda_2 \triangle_2$ as $\lambda_2 \to \infty$ and $\triangle_2 \to 0$, the equations of motion for the imcompressible layer are

$$\left.\begin{aligned} -\frac{\partial \pi}{\partial x} + \mu_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) u_2 = \rho_2 \frac{\partial^2 u_2}{\partial t^2}, \\ -\frac{\partial \pi}{\partial z} + \mu_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) w_2 = \rho_2 \frac{\partial^2 w_2}{\partial t^2}. \end{aligned}\right\} \qquad ...(22)$$

Using the substitutions for u_2 and w_2 as given by (3) and putting

$$-\pi = \rho_2 \frac{\partial^2 \varphi_2}{\partial t^2} \qquad ...(23)$$

we easily obtain the solutions for φ_2 and ψ_2 in this case as

$$\left.\begin{aligned} \varphi_2 &= (A_2 e^{-kz} + A_2' e^{kz}) \cos k(x - ct), \\ \psi_2 &= (B_2 e^{-ks_2 z} + B_2' e^{ks_2 z}) \sin k(x - ct). \end{aligned}\right\} \qquad ...(24)$$

The stress-components $(zz)_2$ and $(\widehat{zx})_2$ in this case are given by

$$\left.\begin{aligned} (\widehat{zz})_2 &= -\pi + 2\mu_2 \frac{\partial w_2}{\partial z} \\ &= \rho_2 \frac{\partial^2 \varphi_2}{\partial t^2} + 2\mu_2\left(\frac{\partial^2 \varphi_2}{\partial z^2} - \frac{\partial^2 \psi_2}{\partial z \partial x}\right) \\ \text{and} \\ (\widehat{zx})_2 &= \mu_2\left(2\frac{\partial^2 \varphi_2}{\partial z \partial x} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{\partial^2 \psi_2}{\partial x^2}\right). \end{aligned}\right\} \qquad ...(25)$$

From the boundary conditions (10) we have in this case

$$A_1 e^{-kn_1 h} + B_1 s_1 e^{-ks_1 h} - A_2 e^{-kh} - A_2' e^{kh} - B_2 s_2 e^{-ks_2 h} + B_2' s_2 e^{ks_2 h} = 0, \qquad ...(26)$$

$$A_1 n_1 e^{-kn_1 h} + B_1 e^{-ks_1 h} - A_2 e^{-kh} - A_2' e^{kh} - B_2 e^{-ks_2 h} - B_2' e^{ks_2 h} = 0, \qquad ...(27)$$

$$A_1 l_1 e^{-kn_1 h} + B_1 2\mu_1 s_1 e^{-ks_1 h} + A_2 (\rho_2 c^2 - 2\mu_2) e^{-kh} + A_2' (\rho_2 c^2 - 2\mu_2) e^{kh} - B_2 2\mu_2 s_2 e^{-ks_2 h} + B_2' 2\mu_2 s_2 e^{ks_2 h} = 0, \qquad ...(28)$$

$$A_1 \frac{2\mu_1}{\mu_2} n_1 e^{-kn_1h} + B_1 \frac{\mu_1}{\mu_2} (1 + s_1^2) e^{-ks_1h}$$
$$- A_2\, 2e^{-kh} + A_2'\, 2e^{kh} - B_2 (1 + s_2^2) e^{-ks_2h}$$
$$- B_2' (1 + s_2^2) e^{ks_2h} = 0, \quad \ldots(29)$$

$$- A_2 - A_2' - B_2 s_2 + B_2' s_2 + A_3 - B_3 s_3 = 0, \quad \ldots(30)$$

$$- A_2 + A_2' - B_2 - B_2' - A_3 n_3 + B_3 = 0, \quad \ldots(31)$$

$$A_2 (\rho_2 c^2 - 2\mu_2) + A_2' (\rho_2 c^2 - 2\mu_2) - B_2 s_2\, 2\mu_2$$
$$+ B_2' s_2\, 2\mu_2 + A_3 l_3 - B_3\, 2\mu_3 s_3 = 0, \quad \ldots(32)$$

$$- 2A_2 + 2A_2' - B_2 (1 + s_2^2) - B_2' (1 + s_2^2)$$
$$- A_3 \frac{2\mu_3}{\mu_2} n_3 + B_3 \frac{\mu_3}{\mu_2} (1 + s_3^2) = 0. \quad \ldots(33)$$

From equations (26), (27), (28), (29), (30), (31), (32) and (33) we obtain the frequency equation (34) (*see page* 384).

Case III

Next let us suppose that the inner layer is composed of material having cubic symmetry, the material in the upper and lower semi-infinite media being compressible as before. Assuming that the axes of symmetry of the cubic material are the axes of reference, the equations of motion for cubic material in two dimensions are

$$c_{11} \frac{\partial^2 u_2}{\partial x^2} + c_{44} \frac{\partial^2 u_2}{\partial z^2} + (c_{12} + c_{44}) \frac{\partial^2 w_2}{\partial x \partial z} = \rho_2 \frac{\partial^2 u_2}{\partial t^2} \quad \ldots(35)$$

$$c_{44} \frac{\partial^2 w_2}{\partial x^2} + c_{11} \frac{\partial^2 w_2}{\partial z^2} + (c_{12} + c_{44}) \frac{\partial^2 u_2}{\partial x \partial z} = \rho_2 \frac{\partial^2 w_2}{\partial t^2} \quad \ldots(36)$$

Substituting

$$\left.\begin{aligned} u_2 &= \frac{\partial \varphi}{\partial x} + \frac{\partial^2 \psi}{\partial x \partial z} \\ \text{and} \quad w_2 &= \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} \end{aligned}\right\} \quad \ldots(37)$$

in equations (35) and (36), we get

$$\frac{\partial}{\partial x} \left[c_{11} \frac{\partial^2 \varphi}{\partial x^2} + (c_{12} + 2c_{44}) \frac{\partial^2 \varphi}{\partial z^2} - \rho_2 \frac{\partial^2 \varphi}{\partial t^2} \right.$$
$$\left. + \frac{\partial}{\partial z} \left\{ c_{11} \frac{\partial^2 \psi}{\partial x^2} + (c_{12} + 2c_{44}) \frac{\partial^2 \psi}{\partial z^2} - \rho_2 \frac{\partial^2 \psi}{\partial t^2} \right\} \right] = 0 \quad \ldots(38)$$

$$\begin{vmatrix} 1 & s_1 & -e^{-kh} & -e^{kh} & -s_2\, e^{-ks_2 h} & s_2\, e^{ks_2 h} & 0 & 0 \\ n_1 & 1 & -e^{-kh} & -e^{kh} & -e^{-ks_2 h} & -e^{ks_2 h} & 0 & 0 \\ l_1 & 2\mu_1 s_1 & (\rho_2 c^2-2\mu_2)\, e^{-kh} & (\rho_2 c^2-2\mu_2)\, e^{kh} & -2\mu_2 s_2\, e^{-ks_2 h} & 2\mu_2 s_2\, e^{ks_2 h} & 0 & 0 \\ 2\dfrac{\mu_1}{\mu_2}\, n_1 & \dfrac{\mu_1}{\mu_2}(1+s_1^2) & -2e^{-kh} & 2e^{kh} & -(1+s_2^2)\, e^{-ks_2 h} & -(1+s_2^2)\, e^{ks_2 h} & 0 & 0 \\ 0 & 0 & -1 & -1 & -s_2 & s_2 & 1 & -s_3 \\ 0 & 0 & -1 & 1 & -1 & -1 & -n_3 & 1 \\ 0 & 0 & \rho_2 c^2-2\mu_2 & \rho_2 c^2-2\mu_2 & -2\mu_2 s_2 & 2\mu_2 s_2 & l_3 & -2\mu_3 s_3 \\ 0 & 0 & -2 & 2 & -(1+s_2^2) & -(1+s_2^2) & -2\dfrac{\mu_3}{\mu_2} n_3 & \dfrac{\mu_3}{\mu_2}(1+s_3^2) \end{vmatrix} = 0 \qquad \ldots(34)$$

and

$$\frac{\partial}{\partial z}\left[c_{11}\frac{\partial^2\varphi}{\partial z^2}+(c_{12}+2c_{44})\frac{\partial^2\varphi}{\partial x^2}-\rho_2\frac{\partial^2\varphi}{\partial t^2}\right]$$
$$+\frac{\partial^2}{\partial z^2}\left[c_{11}\frac{\partial^2\psi}{\partial z^2}+(c_{12}+2c_{44})\frac{\partial^2\psi}{\partial x^2}-\rho_2\frac{\partial^2\psi}{\partial t^2}\right]=0. \quad \ldots(39)$$

Assuming

$$\left.\begin{aligned}\varphi &= M\,e^{-qz}\cos k(x-ct),\\ \psi &= N\,e^{-qz}\cos k(x-ct),\end{aligned}\right\} \quad \ldots(40)$$

we find that equations (38) and (39) are satisfied if $M - Nq = 0$ and q satisfies the equation

$$\{(c_{12}+2c_{44})\,q^2+k^2(\rho_2c^2-c_{11})\}\{c_{11}q^2+k^2(\rho_2c^2-c_{12}-2c_{44})\}=0. \quad \ldots(41)$$

If $\pm\, q_1$ and $\pm\, q_2$ be the roots of equation (41), we may write

$$\left.\begin{aligned}\varphi &= (M_1\,e^{q_1z}+M_2\,e^{-q_1z}+M_3\,e^{q_2z}+M_4\,e^{-q_2z})\cos k(x-ct)\\ \psi &= (N_1\,e^{q_1z}+N_2\,e^{-q_1z}+N_3\,e^{q_2z}+N_4\,e^{-q_2z})\cos k(x-ct)\end{aligned}\right\} \quad \ldots(42)$$

where

$$\left.\begin{aligned}q_1N_1 &= M_1\\ q_1N_2 &= -\,M_2\\ q_2N_3 &= M_3\\ q_2N_4 &= -\,M_4.\end{aligned}\right\} \quad \ldots(43)$$

Hence from (37), (42) and (43) we get

$$\left.\begin{aligned}u_2 &= -\,2k\sin k(x-ct)\,[M_1\,e^{q_1z}+M_2\,e^{-q_1z}\\ &\qquad +M_3\,e^{q_2z}+M_4\,e^{-q_2z}]\,,\\ w_2 &= 2\cos k(x-ct)\,[q_1M_1\,e^{q_1z}-q_1M_2\,e^{-q_1z}\\ &\qquad +q_2M_3\,e^{q_2z}-q_2M_4\,e^{-q_2z}]\,.\end{aligned}\right\} \quad \ldots(44)$$

From the stress-strain relations, we have in this case

$$\left.\begin{aligned}\left(\widehat{zz}\right)_2 &= c_{12}\frac{\partial u_2}{\partial x}+c_{11}\frac{\partial w_2}{\partial z}\\ \left(\widehat{zx}\right)_2 &= c_{44}\left(\frac{\partial u_2}{\partial z}+\frac{\partial w_2}{\partial x}\right)\end{aligned}\right\} \quad \ldots(45)$$

From the boundary conditions (10), we therefore have

$$A_1\, e^{-kn_1h} + B_1s_1\, e^{-ks_1h} - M_1\, 2e^{q_1h} - M_2\, 2e^{-q_1h} - M_3\, 2e^{q_2h} - M_4\, 2e^{-q_2h} = 0 \quad ...(46)$$

$$A_1kn_1\, e^{-kn_1h} + B_1k\, e^{-ks_1h} + M_1\, 2q_1\, e^{q_1h} - M_2\, 2q_1\, e^{-q_1h} + M_3\, 2q_2\, e^{q_2h} - M_4\, 2q_2\, e^{-q_2h} = 0 \quad ...(47)$$

$$A_1k^2\, l_1\, e^{-kn_1h} + B_1\, 2k^2\, \mu_1s_1\, e^{-ks_1h} - M_1\, 2\xi_1\, e^{q_1h} - M_2\, 2\xi_1\, e^{-q_1h} - M_3\, 2\xi_2\, e^{q_2h} - M_4\, 2\xi_2\, e^{-q_2h} = 0 \quad ...(48)$$

$$A_1\, 2k\, \mu_1n_1\, e^{-kn_1h} + B_1\, k\mu_1(1 + s_1^2)\, e^{-ks_1h} + M_1\, 4c_{44}q_1\, e^{q_1h} - M_2\, 4c_{44}q_1\, e^{-q_1h} + M_3\, 4c_{44}q_2\, e^{q_2h} - M_4\, 4c_{44}q_2\, e^{-q_2h} = 0 \quad ...(49)$$

$$2M_1 + 2M_2 + 2M_3 + 2M_4 - A_3 + B_3s_3 = 0 \quad ...(50)$$

$$M_1\, 2q_1 - M_2\, 2q_1 + M_3\, 2q_2 - M_4\, 2q_2 - A_3kn_3 + B_3k = 0 \quad ...(51)$$

$$M_1\, 2\xi_1 + M_2\, 2\xi_1 + M_3\, 2\xi_2 + M_4\, 2\xi_2 - A_3l_3k^2 + B_3\, 2\mu_3s_3k^2 = 0 \quad ...(52)$$

$$M_1\, 4c_{44}q_1 - M_2\, 4c_{44}q_1 + M_3\, 4c_{44}q_2 - M_4\, 4c_{44}q_2 - A_3\, 2k\mu_3n_3 + B_3k\mu_3(1 + s_3^2) = 0 \quad ...(53)$$

where

$$\left.\begin{aligned} \xi_1 &= q_1^2c_{11} - k^2c_{12}\,, \\ \xi_2 &= q_2^2c_{11} - k^2c_{12}\,. \end{aligned}\right\} \quad ...(54)$$

From equations (46), (47), (48), (49), (50), (51), (52) and (53) we obtain the frequency equation (55) (*see page* 387).

Numerical Results

For numerical work we have taken (Bullen 1947)

$$\lambda_1 = 9.0 \times 10^{11}\,, \quad \lambda_2 = 10.1 \times 10^{11}\,, \quad \lambda_3 = 11.4 \times 10^{11}$$
$$\mu_1 = 7.4 \times 10^{11}\,, \quad \mu_2 = 8.1 \times 10^{11}\,, \quad \mu_3 = 9.0 \times 10^{11}$$

all expressed in dynes/cm², and

$$\left.\begin{aligned} \rho_1 &= 3.47 \\ \rho_2 &= 3.53 \\ \rho_3 &= 3.64 \end{aligned}\right\} \text{gm/cm}^3.$$

$$\begin{vmatrix} 1 & s_1 & -2\,e^{q_1h} & -2\,e^{-q_1h} & -2\,e^{q_2h} & -2\,e^{-q_2h} & 0 & 0 \\ kn_1 & k & 2q_1\,e^{q_1h} & -2q_1\,e^{-q_1h} & 2q_2\,e^{q_2h} & -2q_2\,e^{-q_2h} & 0 & 0 \\ k^2 l_1 & 2k^2\mu_1 s_1 & -2\xi_1\,e^{q_1h} & -2\xi_1\,e^{-q_1h} & -2\xi_2\,e^{q_2h} & -2\xi_2\,e^{-q_2h} & 0 & 0 \\ 2k\mu_1 n_1 & k\mu_1(1+s_1^2) & 4c_{44}q_1\,e^{q_1h} & -4c_{44}q_1\,e^{-q_1h} & 4c_{44}q_2\,e^{q_2h} & -4c_{44}q_2\,e^{-q_2h} & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & -1 & s_3 \\ 0 & 0 & 2q_1 & -2q_1 & 2q_2 & -2q_2 & -kn_3 & k \\ 0 & 0 & 2\xi_1 & 2\xi_1 & 2\xi_2 & 2\xi_2 & -k^2 l_3 & 2k^2\mu_3 s_3 \\ 0 & 0 & 4c_{44}q_1 & -4c_{44}q_1 & 4c_{44}q_2 & -4c_{44}q_2 & -2k\mu_3 n_3 & k\mu_3(1+s_3^2) \end{vmatrix} = 0 \quad \ldots(55)$$

It is found that three of the roots of the frequency eqn. (20) are

$$\frac{c}{\beta_2} = 0, 1, \left(\frac{\lambda_2}{\mu_2} + 2\right)^{1/2}$$

that is, $$\frac{c}{\beta_2} = 0, 1, 1.79$$

which are independent of k and hence corresponding to these roots there is no dispersion of the general wave form. However, the dependence of c/β_2 on k as derived from the frequency equation is shown in the following Table I.

Frequency equation (34) readily gives two of its roots as

$$\frac{c}{\beta_2} = 0, 1.$$

TABLE I

$\frac{2\pi}{kh}$	0	1	2	4	6	8
$\frac{c}{\beta_2}$	0.62	0.80	0.87	0.91	0.95	0.96

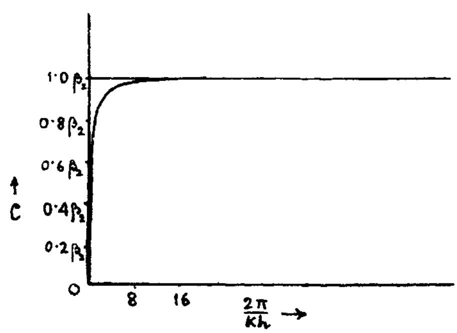


FIG. 1

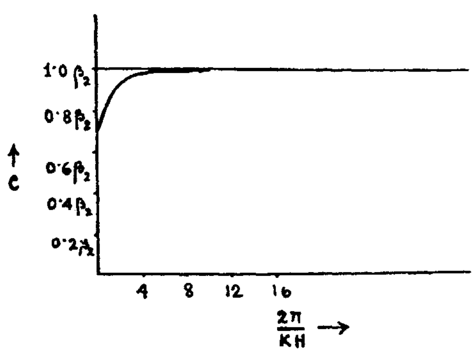


FIG. 2

Three of the roots of the frequency eqn. (55) are also easily found to be

$$\frac{c}{\beta_2}=0,\left(2+\frac{c_{12}}{c_{44}}\right)^{1/2},\ \left(\frac{c_{11}}{c_{44}}\right)^{1/2}.$$

Taking the values of the elastic constants for cubic material to be those for Pyrites (Love 1944, p. 163) we get in the last case

$$\frac{c}{\beta_2}=0,\ 1.24,\ 2.03$$

The variation of c against $2\pi/kh$ for Case I is represented graphically in Fig. 1. The curve in Fig. 2 represents the corresponding variation obtained by Sezawa and Nishimura for a model of similar type. The result thus obtained shows that Rayleigh-type waves propagated along an inner stratum are dispersive in nature. The variation of c against $2\pi/kh$ as obtained by the authors are very much similar to that obtained by Sezawa and Nishimura.

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