SOME RESULTS ON FIXED POINTS

by BARADA K. RAY, *Department of Mathematics, Regional Engineering College, Durgapur* 713209

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Some fixed point theorems related to a result of Diaz and Metcalf (1969) and a theorem of Petryshyn (1971) have been presented in this paper.

§1. Let $T : X \to X$ be a continuous mapping of a metric space (X, d) into itself. We introduce the following definitions.

Definition 1.1 (Kuratowski 1958)—Let $A \subset X$ be a bounded set. By the real number $\alpha(A)$ we denote the infimum of all numbers $\epsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ϵ.

It is easily seen that

(i) $\alpha(A) \leqslant \delta(A)$, where $\delta(A)$ is the diameter of the set $A \subset X$;

(ii) $\alpha(A) = 0$ if and only if A is precompact;

(iii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$

(iv) $\alpha(A + B) \leqslant \alpha(A) + \alpha(B)$, where A, B are subsets of a (normed) linear space.

(v) $\alpha(\bar{A}) = 0 \Leftrightarrow \alpha(A) = 0$ (see Szufla 1968) where $\bar{A}$ is the closure of A.

Definition 1.2 (see Furi and Vignoli 1970)—The continuous mapping T is called condensing, if for every bounded subset A of X such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$. Obviously contractive mappings and completely continuous mappings are condensing.

We need the following two theorems :

Theorem 1.3 (see Furi and Vignoli 1970)—Let $T : K \to K$ be a condensing mapping defined on a closed bounded convex subset K of a Banach space X. Then T has at least one fixed point.

Theorem 1.4 (see Theorem 3, page 474 of Diaz and Metcalf 1969)—Let $T : X \to X$ be a continuous mapping of a nonempty metric space X into itself.

Suppose

(1) $F(T)$ is nonempty, where $F(T)$ is the set of fixed points of T;

(2) for each $x \in X$, with $x \notin F(T)$, and each $p \in F(T)$ one has $d(Tx, p) < d(x, p)$.

Let $x \in X$. Then, either $\left\{ T^n x \right\}_{n=0}^{\infty}$ contains no convergent subsequence, or $\lim_{n \to \infty} T^n x$ exists and belongs to $F(T)$.

We prove the following theorem :

Theorem 1.5—Let $T : K \to K$ be a condensing mapping defined on a closed bounded convex subset K of a strictly convex Banach space X. Let T satisfy the following condition

$$\| Tx - Ty \| \leqslant a \| x - y \| + b \{\| x - Tx \| + \| y - Ty \|\} + c \{ \| x - Ty \| + \| y - Tx \| \} \quad \text{...(1.5.1)}$$

for all $x, y \in K$ and for positive a, b, c with $a + 2b + 2c \leqslant 1$. Then for each x in K, the Picard sequence, starting from x and generated by the transformation T_λ, where

$$T_\lambda x = \lambda x + (1 - \lambda) Tx, \; 0 < \lambda < 1, \quad \text{...(1.5.2)}$$

converges to a fixed point of T.

PROOF : Obviously T_λ is defined on K and since K is convex, $T_\lambda K \subset K$. Further T_λ is condensing. Indeed, let A be a bounded non-precompact subset of K, then $T_\lambda A = \lambda A + (1 - \lambda) TA$.

Hence $\alpha (T_\lambda A) \leqslant \lambda \alpha (A) + (1 - \lambda) \alpha (TA)$

$$< \lambda \alpha (A) + (1 - \lambda) \alpha (A) = \alpha (A).$$

It is obvious that $F(T)$ and $F(T_\lambda)$ coincide for every λ; and by Theorem 1.3 $F(T)$ (and hence $F(T_\lambda)$) is nonempty.

For $x \in K$, let $A = U_{n=0}^{\infty} T_\lambda^n x$.

We have $T_\lambda A = U_{n=1}^{\infty} T_\lambda^n x$.

Then A is an invariant set; $A = \{x\} \, U \, T_\lambda A$.

Denote by $\bar{A}$ the closure of A. $\bar{A}$ is an invariant set too; indeed, from the continuity of T_λ, it follows :

$$T_\lambda \bar{A} \subset \overline{T_\lambda A} \subset \bar{A}.$$

Next we show that $\bar{A}$ is compact. For this it is sufficient to show that $\alpha(A) = 0$, since in a complete metric space (and therefore in a Banach space) the precompact sets are also relatively compact.

Suppose $\alpha(A) > 0, \quad A = \{x\} \cup T_\lambda A.$

$$\text{Then } \alpha(A) = \max\{\alpha(T_\lambda A), \alpha(x)\}$$
$$= \max\{\alpha(T_\lambda A), 0\}$$
$$= \alpha(T_\lambda A).$$

But this contradicts the fact that T_λ is condensing. Hence $\alpha(A) = 0$ and $\bar{A}$ is compact. Hence the sequence of iterates has a convergent subsequence. Now let $p \in F(T)$, hence $p \in F(T_\lambda)$ and let $x \in K - F(T)$. Then from (1.5.1) we have

$$\|Tx - p\| = \|Tx - Tp\| \leqslant a\|x - p\| + b\{\|x - Tx\| + \|p - Tp\|\}$$
$$+ c\{\|x - Tp\| + \|p - Tx\|$$
$$\leqslant a\|x - p\| + b\|x - p\| + b\|Tx - p\|$$
$$+ c\|x - p\| + c\|Tp - p\|$$
$$+ c\|Tx - p\|$$
$$= a\|x - p\| + b\|x - p\| + c\|x - p\|$$
$$+ b\|Tx - p\| + c\|Tx - p\|$$

i.e. $\|Tx - p\| \leqslant \dfrac{a + b + c}{1 - b - c}\|x - p\| \leqslant \|x - p\|$

But X is strictly convex. So $\|Tx - p\| < \|x - p\|$

and

$$\|T_\lambda x - p\| = \|T_\lambda x - T_\lambda p\|$$
$$= \|\lambda x + (1 - \lambda)Tx - \lambda p - (1 - \lambda)Tp\|$$
$$\leqslant \lambda\|x - p\| + (1 - \lambda)\|Tx - Tp\|$$
$$< \lambda\|x - p\| + (1 - \lambda)\|x - p\| = \|x - p\|.$$

Hence by Theorem 1.4, $\left\{T_\lambda^n x\right\}$ converges to a fixed point of T.

The following theorem follows from Theorem 1.5 as a Corollary.

Theorem 1.6—Let K be a closed subset of a strictly convex Banach space X and let $T : K \to K$ be a continuous transformation which satisfies (1.5.1). If $T(K)$ is contained in a compact subset K_1 of K, then, for every x in K, the Picard sequence starting from x and generated by T_λ defined by (1.5.2) converges to a fixed point of T.

PROOF : As in Theorem 1.5, $F(T) = F(T_\lambda)$ and by Schauder's (1930) theorem, $F(T) \neq \phi$, then $F(T_\lambda) \neq \phi$. Now, since $T(K)$ is contained in a compact subset K_1 of K, $\alpha(T(K)) = 0$ i.e. T is completely continuous and hence condensing.

Therefore, since X is strictly convex and T satisfies (1.5.1) we have, for every $y \in K - F(T)$ and $u \in F(T)$,

$$\| T_\lambda y - u \| < \| y - u \|$$

Definition 1.7—Let X be a Banach space. A mapping T of X into itself is called nonexpansive if

$$\| Tx - Ty \| \leqslant \| x - y \| \ \forall \ x, y \in X.$$

Theorem 4 of Diaz and Metcalf (1969) can be derived from Theorem 1.5 as a Corollary.

Corollary 1.8—Let K be a closed convex subset of a strictly convex Banach space X; $T : K \to K$ a continuous nonexpansive mapping defined in K such that $T(K)$ is a relatively compact set contained in K. Let $T_\lambda = \lambda I + (1 - \lambda)T$, $0 < \lambda < 1$.

Then for each x_0 in K, the sequence $\left\{ T_\lambda^n x_0 \right\}$ converges to a fixed point of T.

Corollary 1.9—(see Petryshyn 1971)—Let X be a strictly convex Banach, K a closed bounded convex subset of X, and $T : K \to K$ a condensing nonexpansive mapping.

For each λ, $0 < \lambda < 1$, let $T_\lambda = \lambda I + (1 - \lambda)T$.

Then for each x_0 in K, the sequence $\{x_{n+1}\} = \left\{ T_\lambda^n x_0 \right\}$ determined by the iteration method

$$x_{n+1} = \lambda x_n + (1 - \lambda) Tx_n, \ n = 0, 1, 2, \ldots, \ x_0 \in K$$

converges to a fixed point of T.

PROOF: T is nonexpansive and condition (1.5.1) is satisfied with $a = 1$, $b = c = 0$. Since T_λ is nonexpansive and X is strictly convex, it follows easily that

$$\| T_\lambda y_0 - u \| < \| y_0 - u \| \text{ whenever } u \in F(T), y_0 \in K - F(T).$$

§2. Nonlinear functional equations in Banach space: Let X be a Banach space, T a (possibly) nonlinear selfmapping of X. We are concerned with the solvability of the equation

$$u - Tu = f \qquad \ldots(2.1)$$

for a given element $f \in X$ and its relation to the properties of the Picard iterates for the equation (2.1), i.e. the sequence $\{x_n\}$ where

$$x_{n+1} = Tx_n + f, \ \ n = 0, 1, \ldots, x_0 \text{ given.} \qquad \ldots(2.2)$$

We need the following definitions and theorems.

Definition 2.1—A mapping $T : X \to X$ is said to be asymptotically regular if for each $x \in X$, $T^{n+1} x - T^n x \to 0$ strongly in X as $n \to \infty$, T is said to be weakly asymptotically regular if $T^{n+1} x - T^n x \to 0$ weakly in X for each $x \in X$ as $n \to \infty$.

Theorem 2.2—(see Petrysyn and Browder 1966). Let T be a nonlinear nonexpansive mapping of X into itself and suppose X is uniformly convex. Then the equation (2.1) has a solution u for a given $f \in X$ if and only if for any specific $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ starting from x_0 is bounded in X.

Theorem 2.3 (see Petrysyn and Browder 1966)—Let T be a nonexpansive mapping of X into itself. For a given f in X, let

$$T_f\, u = Tu + f,$$

and suppose T_f is weakly asymptotically regular. Let $x_n = T_f^n\, x_0$ be the sequence of Picard iterates for the equation (2.01) starting with x_0 and suppose that an infinite subsequence of $\{x_n\}$ converges strongly to an element y of X. Then y is a solution of (2.1) and the whole sequence $\{x_n\}$ converges strongly to y.

We prove the following theorem.

Theorem 2.4—Let T be a nonexpansive self mapping of a uniformly convex Banach space X such that T has a nonempty set $F(T)$ of fixed points. For a given constant λ, with $0 < \lambda < 1$, let

$$T_\lambda = \lambda\, T + (1 - \lambda)\, I.$$

Then T_λ is asymptotically regular and has the same fixed points as T. Hence the fixed points of T can be obtained from iteration of T_λ for which conclusions of Theorems 2.2–2.3 can be applied.

PROOF : Obviously T_λ is nonexpansive and $F(T)$ and $F(T_\lambda)$ coincide. Let $u \in F(T)$ and for a given $x_0 \in X$, let $x_n = T_\lambda^n\, x_0$. Since T_λ is nonexpansive and u is a fixed point of T_λ we have

$$\| x_{n+1} - u \| \leqslant \| x_n - u \| \text{ for all } n$$

and hence $\| x_n - u \|$ converges to a nonnegative limit η.

Suppose $\eta > 0$. Since $x_{n+1} - u = \lambda\, (Tx_n - Tu) + (1 - \lambda)\, (x_n - u)$ and since $\| x_n - u \| \to \eta$, $\| x_{n+1} - u \| \to \eta$, $\| Tx_n - Tu \| \leqslant \| x_n - u \|$, it follows from the uniform convexity of X that $\| x_n - u - (Tx_n - u) \| \to 0$ *i.e.* $x_n - Tx_n \to 0$ strongly in X. Hence $x_{n+1} - x_n \to 0$ strongly in X i.e. T_λ is asymptotically regular.

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