ON H-CURVATURE TENSORS IN ALMOST PRODUCT AND ALMOST DECOMPOSABLE MANIFOLD II

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The recurrence and other properties of the H-curvature tensors in an almost product and almost decomposable manifold have been studied. The condition that an almost product and almost decomposable manifold is an almost Einstein manifold has been obtained.

1. INTRODUCTION

We consider an n-dimensional manifold M_n of differentiability class C^∞ endowed with a real vector valued linear function F such that for arbitrary vector fields X, Y, Z ... in M_n

$$\bar{\bar{X}} = X, \qquad ...(1.1)$$

If in M_n, there is a positive definite Riemannian metric g such that

$$g(\bar{X}, \bar{Y}) = g(X, Y), \qquad ...(1.2)$$

then M_n is said to be an almost product manifold. Let D be a Riemannian connexion in M_n. Then if

$$(D_X F)\, Y = 0, \qquad ...(1.3)$$

we say that M_n is an almost product and almost decomposable manifold (Mishra 1970).

Let K be the curvature tensor in M_n then

$$K(X, Y, \bar{Z}) = \overline{K(X, Y, Z)} \qquad ...(1.4a)$$

$$K(X, Y, Z) + K(Y, X, Z) = 0 \qquad ...(1.4b)$$

$$'K(X, Y, Z, U) = {'K}(X, \bar{Y}, \bar{Z}, \bar{U}) \qquad ...(1.4c)$$

$$\text{Ric}\,(\bar{X}, \bar{Y}) = \text{Ric}\,(X, Y) \qquad ...(1.4d)$$

$$(D_{FX}\,\text{Ric})\,(\bar{Y}, Z) = (D_Z\,\text{Ric})\,(X, Y) - (D_Y\,\text{Ric})\,(X, Z) \qquad ...(1.4e)$$

$$X.R = 2\,(\text{div}\; r)\,(X) \qquad ...(1.4f)$$

$$K(X, Y, Z) + K\,(Y, Z, X) + K(Z, X, Y) = 0 \qquad ...(1.4g)$$

where Ric, r are the Ricci tensors of type (0, 2) and (1, 1), R the scalar curvature and $'K(X, Y, Z, U) = g(K(X, Y, Z) U)$ (Mishra 1970).

An almost product and almost decomposable manifold is said to be an almost Einstein manifold if (Yano 1965)

$$\text{Ric}(X, Y) = ag(X, Y) + b'F(X, Y) \quad \ldots(1.5)$$

where $'F(X, Y) = g(\bar{X}, Y)$ and a, b are constants.

An almost product and almost decomposable manifold is called a manifold of almost constant curvature if (Yano 1965)

$$\begin{aligned} 'K(X, Y, Z, U) &= a\{g(X, U) g(Y, Z) - g(Y, U) g(X, Z) + 'F(X, U)\,'F(Y, Z) \\ &\quad - 'F(Y, U)\,'F(X, Z)\} + b\{'F(X, U) g(Y, Z) - 'F(Y, U) \\ &\quad g(X, Z) + g(X, U)\,'F(Y, Z) - g(Y, U)\,'F(X, Z)\}. \end{aligned} \quad \ldots(1.6)$$

H-projective curvature tensor P, H-conformal curvature tensor Q, H-conharmonic curvature tensor S and H-concircular curvature tensor T in almost product and almost decomposable manifold are given by (Sinha and Singh 1975)

$$\begin{aligned} P(X, Y, Z) &= K(X, Y, Z) + \frac{1}{n-2}\Big\{\text{Ric}(X, Z) Y - \text{Ric}(Y, Z) X \\ &\quad + \text{Ric}(\bar{X}, Z)\bar{Y} - \text{Ric}(\bar{Y}, Z)\bar{X}\Big\} \end{aligned} \quad \ldots(1.7)$$

$$\begin{aligned} Q(X, Y, Z) &= K(X, Y, Z) + \frac{1}{n-4}\Big\{\text{Ric}(X, Z) Y - \text{Ric}(Y, Z) X \\ &\quad + \text{Ric}(\bar{X}, Z)\bar{Y} - \text{Ric}(\bar{Y}, Z)\bar{X} + r(Y) g(X, Z) - r(X) g(Y, Z) \\ &\quad + r(\bar{Y}) g(\bar{X}, Z) - r(\bar{X}) g(\bar{Y}, Z)\Big\} - \frac{R}{(n-2)(n-4)}\Big\{g(X, Z) Y \\ &\quad - g(X, Z) X + g(\bar{X}, Z)\bar{Y} - g(\bar{Y}, Z)\bar{X}\Big\} \end{aligned} \quad \ldots(1.8)$$

$$\begin{aligned} S(X, Y, Z) &= K(X, Y, Z) + \frac{1}{n-4}\Big\{\text{Ric}(X, Z) Y - \text{Ric}(Y, Z) X \\ &\quad + \text{Ric}(\bar{X}, Z)\bar{Y} - \text{Ric}(Y, Z)\bar{X} + r(Y) g(X, Z) - r(X) g(Y, Z) \\ &\quad + r(\bar{Y}) g(\bar{X}, Z) - r(\bar{X}) g(\bar{Y}, Z)\Big\} \end{aligned} \quad \ldots(1.9)$$

$$\begin{aligned} T(X, Y, Z) &= K(X, Y, Z) + \frac{R}{n(n-2)}\Big\{g(X, Z) Y - g(Y, Z) X - g(\bar{X}, Z)\bar{Y} \\ &\quad - g(\bar{Y}, Z)\bar{X}\Big\}. \end{aligned} \quad \ldots(1.10)$$

2. *H*-Curvature Tensors

The following identities are obtained by simple computations :

$$B(X, Y, Z) = \overline{B(\overline{X}, \overline{Y}, \overline{Z})}$$

$$'B(X, Y, \overline{Z}, \overline{U}) = 'B(\overline{X}, \overline{Y}, Z, U) = 'B(X, Y, Z, U)$$

$$'B(X, Y, \overline{Z}, U) = 'B(X, Y, Z, \overline{U})$$

where B stands for P, Q, S or T.

Theorem 2.1—In almost product and almost decomposable manifold M_n, we have

$$(C_3^1\, B)\,(X, Y) = 0 \qquad ...(2.1)$$

$$(C_1^1\, P)\,(Y, Z) = -f\,\mathrm{Ric}\,(\overline{Y}, Z)/(n-2) \qquad ...(2.2)$$

$$(C_1^1\, Q)\,(Y, Z) = \frac{1}{n-4}\left\{R\,fg(Y, Z)/(n-2) - \tilde{f}\,g(Y, Z) - f\,\mathrm{Ric}\,(\overline{Y}, Z)\right\} \qquad ...(2.3)$$

$$(C_1^1\, S)\,(Y, Z) = \frac{1}{n-4}\left\{f\,\mathrm{Ric}\,(\overline{Y}, Z) + R\,g(Y, Z) + \tilde{f}\,g(\overline{Y}, Z)\right\} \qquad ...(2.4)$$

$$(C_1^1\, T)\,(Y, Z) = \mathrm{Ric}\,(Y, Z) - \frac{R}{n}\,g(Y, Z) - \frac{R\,f}{n(n-2)}\,'F(Y, Z), \qquad ...(2.5)$$

where $f \overset{def}{=} (C_1^1\, F)$, C_1^1 and C_3^1 denote contractions of first and third slots respectively.

Proof : Proof is obvious from (1.7)-(1.10).

Theorem 2.2—In an almost Einstein manifold scalar curvature is constant and consequently f is constant.

Proof : In an almost Einstein manifold, we have

$$r(X) = aX + b\overline{X}. \qquad ...(2.6)$$

Covariant derivative of (2.6) gives

$$(D_Y\, r)\,(X) = 0$$

equivalently $(\mathrm{div}\; r)\,(X) = \frac{1}{2}\,X.R = 0$ gives R as constant where R is scalar curvature. From (2.6), we have

$$R = na + bf,$$

which by virtue of first part gives the proof of the second part of this theorem.

Theorem 2.3—An almost product and almost decomposable manifold M_n of almost constant curvature is an almost Einstein manifold provided f is constant.

PROOF : Let M_n be of an almost constant curvature. Then from (1.6), we get

$$\text{Ric}\,(Y, Z) = (a(n-2) + bf)\, g(Y, Z) + (b(n-2) + af)\, 'F(X, Z).$$

Thus if f is constant then it satisfies (1.5) and hence M_n is an almost Einstein manifold.

Theorem 2.4—An almost Einstein manifold M_n is H-projectively flat, H-conformally flat or H-conharmonically flat if and only if it is of an almost constant curvature.

PROOF : From (1.5) and (1.7) we see that if an almost Einstein manifold M_n is H-projectively flat then it is of an almost constant curvature.

Conversely, if an almost Einstein manifold M_n is of an almost constant curvature then from (1.6) we see that it is H-projectively flat. The proof of the remaining parts of the statement follows the same pattern.

Combining Theorems 2.3 and 2.4, we have

Theorem 2.5—An almost product and almost decomposable manifold of an almost constant curvature is H-projectively flat, H-conformally flat or H-conharmonically flat provided f is constant.

Remark : If M_n is H-projectively flat then from (2.2) we see that f vanishes. Also we see that the Riemannian curvature K is linear combination of X, Y, $\bar{X}$ and $\bar{Y}$.

Theorem 2.6—If the Riemannian curvature K in almost a product and almost decomposable manifold M_n is linear combination of X, Y, $\bar{X}$ and $\bar{Y}$ then it is H-projectively flat provided f vanishes.

PROOF : Let K be a linear combination of X, Y, $\bar{X}$ and $\bar{Y}$ then there must exist four tensors α, β, γ, δ of type (0, 2) such that

$$K(X, Y, Z) = \alpha(Y, Z)\, X + \beta(X, Z)\, Y + \gamma(X, Z)\, \bar{X} + \delta(X, Z)\, \bar{Y}. \quad \ldots(2.7)$$

From (2.7) and (1.4), we have

$$\{\alpha(Y, \bar{Z}) - \gamma(Y, Z)\}\, X + \{\beta(X, \bar{Z}) - \delta\,(X, Z)\}\, Y$$
$$+ \{\gamma(Y, \bar{Z}) - \alpha(Y, Z)\}\, \bar{X} + \{\delta(X, \bar{Z}) - \beta(X, Z)\}\, \bar{Y} = 0.$$

This is true for arbitrary vector fields X, Y and Z, so we have

$$\alpha(Y, \bar{Z}) - \gamma(Y, Z) = 0 \text{ and } \beta(X, \bar{Z}) - \delta(X, Z) = 0. \quad \ldots(2.8)$$

Now, (2.7) with the help of (1.4*b*) gives

$$\{\alpha(Y, Z) + \beta(Y, Z)\} X + \{\beta(X, Z) + \alpha(X, Z)\} Y + \{\gamma(Y, Z) + \delta(Y, Z)\} \bar{X} + \{\gamma(X, Z) + \delta(X, Z)\} \bar{Y} = 0.$$

This is also true for arbitrary vector fields X, Y and Z. So we have

$$\alpha(X, Z) + \beta(X, Z) = 0 \text{ and } \gamma(X, Z) + \delta(X, Z) = 0. \quad ...(2.9)$$

From (2.8) and (2.9), we get

$$\alpha(X, \bar{Z}) = -\beta(X, \bar{Z}) = -\delta(X, Z) = \gamma(X, Z) \quad ...(2.10)$$

By virtue of (2.10) (2.7) becomes

$$K(X, Y, Z) = \gamma(Y, \bar{Z}) X - \gamma(X, \bar{Z}) Y + \gamma(Y, Z) \bar{X} - \gamma(X, Z) \bar{Y}. \quad ...(2.11)$$

Contracting (2.11) in first slot, we have

$$\text{Ric}(Y, Z) = n\,\gamma(Y, \bar{Z}) - \gamma(Y, \bar{Z}) + \gamma(Y, Z) f - \gamma(\bar{Y}, Z). \quad ...(2.12)$$

Using (1.4*d*) in (2.12), we get

$$n\{\gamma(\bar{Y}, Z) - \gamma(Y, \bar{Z})\} + \{\gamma(\bar{Y}, \bar{Z}) - \gamma(Y, Z)\} f = 0$$

which gives

$$\gamma(\bar{Y}, \bar{Z}) = \gamma(Y, Z)$$

if f vanishes. Consequently (2.12) gives

$$\text{Ric}(Y, Z) = (n - 2)\,\gamma(\bar{Y}, Z),$$

which on substitution in (2.11) shows $P = 0$.

Combining Theorem 2.6 and Remark stated above, we have :

Theorem 2.7—An almost product and almost decomposable manifold M_n is H-projectively flat if and only if the Riemannian curvature K is a linear combination of X, Y, $\bar{X}$ and $\bar{Y}$ provided f vanishes.

Symmetric and recurrent properties: An almost product and almost decomposable manifold M_n is said to be H-projective recurrent if

$$(D_X P)(Y, Z, U) = u(X) P(Y, Z, U) \quad ...(2.13)$$

where u is recurrence parameter and M_n is said to be H-projective symmetric if

$$(D_X P)(Y, Z, U) = 0. \quad ...(2.14)$$

Theorem 2.8—An almost product and almost decomposable manifold M_n is H-projective symmetric if and only if it is symmetric.

PROOF : From (1.7) and (1.3), we have

$$(D_U P)(X, Y, Z) = (D_U K)(X, Y, Z) + \{(D_U \text{Ric})(X, Z) Y - (D_U \text{Ric})(Y, Z) X + (D_U \text{Ric})(\bar{X}, Z) \bar{Y} - (D_U \text{Ric})(\bar{Y}, Z)\bar{X}\}/(n-2). \quad ...(2.15)$$

If M_n is H-projective symmetric then (2.15) gives

$$(D_U K)(X, Y, Z) = \{(D_U \text{Ric})(Y, Z) X - (D_U \text{Ric})(X, Z) Y + (D_U \text{Ric})(\bar{Y}, Z) \bar{X} - (D_U \text{Ric})(\bar{X}, Z) \bar{Y}\}/(n-2). \quad ...(2.16a)$$

Contracting X in the above, we have

$$(D_U \text{Ric})(X, Z) = 0. \quad ...(2.16b)$$

Putting (2.16*b*) in (2.16*a*) we see that M_n is symmetric. Converse of the theorem obviously follows from (2.15).

Corollary—In H-projective symmetric manifold M_n, the scalar curvature is constant.

Proof is obvious from (2.16*a*).

Theorem 2.9—An almost product and almost decomposable manifold M_n is H-projective recurrent if and only if it is recurrent for the same recurrence parameter.

PROOF : We know that every recurrent manifold is Ricci recurrent for the same recurrence parameter. Thus if M_n is recurrent then (2.15) with the help of (2.1) gives

$$(D_U P)(X, Y, Z) = u(U) P(X, Y, Z).$$

That is, it is H-projective recurrent.

Conversely, let it be H-projective recurrent then (2.15) gives

$$u(U) P(X, Y, Z) = (D_U K)(X, Y, Z) + \{(D_U \text{Ric})(X, Z) Y - (D_U \text{Ric})(\bar{Y}, Z) X + (D_U \text{Ric})(\bar{X}, Z) \bar{Y} - (D_U \text{Ric})(\bar{Y}, Z) \bar{X}\}/(n-2). \quad ...(2.17a)$$

Contracting X in (2.17*a*) and using (2.2), we get

$$(D_U \text{Ric})(X, Y) = u(U) \text{Ric}(X, Y). \quad ...(2.17b)$$

Putting (2.17*b*) in (2.17*a*) and using (1.7), we get

$$(D_U K)(X, Y, Z) = u(U) K(X, Y, Z).$$

Hence M_n is recurrent.

Theorem 2.10—H-concircularly flat manifold M_n with non-vanishing constant scalar curvature R is an Einstein manifold.

PROOF : Let M_n be H-concircularly flat, then from (1.10) we have

$$\text{Ric}\,(Y, Z) = \frac{R}{n}\; g(Y, Z) + \frac{R\,f}{n\,(n-2)}\, g(\bar{Y}, Z).$$

The above equation yields $R f^2 = 0$. But $R \neq 0$ gives $f = 0$ and hence the above equation reduces to

$$\text{Ric}\,(Y, Z) = \frac{R}{n}\; g(Y, Z).$$

Hence M_n is an Einstein manifold.

REFERENCES

Mishra, R. S. (1970). On almost product and almost decomposable manifolds. *Tensor N. S.*, **21**, 255-60

Yano, K. (1965). Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press.

Sinha, B. B., and Singh, I. J. P. (1975). On *H*-curvature tensors in almost product and almost decomposable manifold, to appear in *Indian J. pure appl. Math.*