MAXIMA OF A RANDOM ALGEBRAIC CURVE

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For the polynomial $\sum_{k=1}^{n} k^p c_k(t)x^n$, where $p \geqslant 0$ and $\{c_k(t)\}$ is a sequence of independent and continuous functions defined on the interval $0 \leqslant t \leqslant 1$ such that for all choice of α_v and β_v the measure of the set of t's for which the relation $\alpha_v \leqslant c_v(t) \leqslant \beta_v$ holds is

$$(2\pi)^{-1/2} \int_{\alpha_v}^{\beta_v} \exp[-(1/2)\, u^2]\, du,$$

we estimate the asymptotic value of the average number of maxima when n is large.

§1. Consider the polynomial

$$y = \sum_{k=1}^{n} k^p c_k(t)\, x^n \qquad \ldots(1.1)$$

where $p \geqslant 0$ and $\{c_k(t)\}$ is a sequence of independent and continuous functions defined on the interval $0 \leqslant t \leqslant 1$ such that for all choice of α_v and β_v, the measure of the set of t's for which the relation $\alpha_v \leqslant c_v(t) \leqslant \beta_v$ holds is

$$(2\pi)^{-1/2} \int_{\alpha_v}^{\beta_v} \exp[-(1/2)\, u^2]\, du.$$

Dunnage (1966) has proved the existence of such functions by using the classical result of Steinhaus. Clearly the c's form a sequence of mutually independent, normally distributed random variables with mean zero and variance one.

Let $M_n(a, b)$ denote the average number of maxima of the curve (1.1) in the interval $a \leqslant x \leqslant b$. Then we want to show that for

$$\delta = \exp[-(\log n)^{1/3}],\ T = (\log n)^{1/2}$$

(i) $M_n(1 - \delta, 1 - T/n) \sim (4\pi)^{-1} [(2p + 3)]^{1/2} \log n,$

(ii) $M_n(1 + T/n, 1 + \delta) \sim (4\pi)^{-1} \log n,$

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(iii) each of $M_n(0, 1 - \delta)$, $M_n(1 - T/n, 1 + T/n)$ and $M_n(1 + \delta, \infty)$ does not exceed a constant multiple of $(\log n)^{1/2}$, and

(iv) $M_n(\alpha, \beta) = M_n(-\beta, -\alpha)$ when $\alpha\beta \geqslant 0$.

In otherwords

$$M_n(-\infty, \infty) \sim (2\pi)^{-1} \{[(2p + 3)]^{1/2} + 1\} \log n,$$

for large n.

When $p = 0$, that is for the polynomial $\sum_{k=1}^{n} c_k(t)\, x^k$, Das (1969) estimated the average number of maxima for large n. For (1.1) Das (1972) estimated the average number of real zeros when n is large.

§2. We find that the polynomial (1.1) is a one parameter family of curves in the xy-plane. From the work of Das (1972) the members of the family on the average will cross the x-axis nearly

$(2\pi)^{-1} \log n$ times when x is outside $-1 \leqslant x \leqslant 1$ and

$(2\pi)^{-1} (2p + 1)^{1/2} \log n$ when x is in $-1 \leqslant x \leqslant 1$, for large n.

Always a random curve will have a number of oscillations determined by its turning points (both maxima and minima) and we may think that the curve oscillates between its points of consecutive maxima (or minima). Hence a curve will have at least half as many oscillations as it has crossings of the x-axis.

From Das (1969) we find that the number of oscillations of the curves $\Sigma\, c_k(t) x^k$ in $-1 \leqslant x \leqslant 1$ is $\sqrt{3}$ times the number indicated by their axis crossings. But it is interesting to note that for the general curves, $\Sigma\, k^p\, c_k(t)\, x^k$, from the result stated in section 1, the number of oscillations is not $\sqrt{3}$ times the number indicated by their axis crossings in $-1 \leqslant x \leqslant 1$.

Since the curve in the range $(-\infty, 0)$, $(0, \infty)$ are the reflections about the y-axis we get (iv) of section 1.

A curve given in (1.1) may have stationary points. If a curve with parameter τ has stationary points at $x = \xi$, then the relations

$$\sum_{k=1}^{n} k^{p+1}\, c_k(\tau)\, \xi^{k-1} = 0$$

$$\sum_{k=2}^{n} k^{p+1}\, (k - 1)\, c_k(\tau)\, \xi^{k-2} = 0$$

necessarily hold. But the totality of such parameter τ forms a set E of measure zero. Therefore let us denote the number of real zeros (counted according to their multiplicities) of the polynomial

$$P(x, t) = \sum_{k=1}^{n} k^{p+1} c_k(t)\, x^{k-1} \qquad \ldots(2.1)$$

in the range $\alpha \leqslant x \leqslant \beta$ as $n(t : \alpha, \beta)$. Hence the foregoing considerations show that

$$M_n(\alpha, \beta) = (1/2)\,[\text{average number of real zeros of (2.1) in } (\alpha, \beta)] + \theta$$

$$= (1/2) \int_0^1 n(t : \alpha, \beta)\, dt + \theta, \qquad \ldots(2.2)$$

where $|\theta| \leqslant 1$ and hence it is real zeros of (2.1) which will be the subject of our further discussion.

§3. Let the n-dim set of points $\bar{c} \equiv (c_1, c_2, \ldots, c_n)$ be denoted by R_n. The probability mass attached to the 'infinitesimal rectangle' $\Pi(\bar{c})$ containing the point $\bar{c}$ is

$$dP(\bar{c}) = \prod_{k=1}^{n} [(2\pi)^{-1/2} \exp(-c_k^2/2)\, dc_k]$$

where $dc_1, \ldots, dc_n$ are the lengths of the sides of $\Pi(\bar{c})$.

Let

$$G_k = \{ t : c_k \leqslant c_k(t) \leqslant c_k + dc_k\},\ k = 1, 2, \ldots, n.$$

Then $m(G_k)$, the measure of the set G_k is

$$(2\pi)^{-1/2} \int_{c_k}^{c_k+dc_k} \exp[-(1/2)\, u^2]\, du$$

$$= (2\pi)^{-1/2} \exp[-(1/2)\,(c_k')^2]\, dc_k$$

by the mean value theorem when $c_k \leqslant c_k' \leqslant c_k + dc_k$. Now following Das (1969, Lemma 1), if $N_n(a, b)$ is the number of real zeros of $P(x, t)$ in the interval $a \leqslant x \leqslant b$ we get

$$E[N_n(a, b)] = \int_0^1 n(t : a, b)\, dt$$

$$= \int\cdots\int_{R_n} n(\bar{c} : a, b)\, dP(\bar{c})$$

$$= \pi^{-1} \int_a^b [(AC - B^2)^{1/2}/A]\, dx \qquad \ldots(3.1)$$

where

$$\left.\begin{aligned} A_p &= A_p(x) = \sum_{k=1}^{n} k^{2p+2}\, x^{2k-2} \\ B_p &= B_p(x) = \sum_{k=2}^{n} k^{2p+2}(k-1)\, x^{2k-3} \\ \text{and} \\ C_p &= C_p(x) = \sum_{k=2}^{n} k^{2p+2}(k-1)^2\, x^{2k-4} \end{aligned}\right\} \qquad \ldots(3.2)$$

provided $AC - B^2 > 0$, which is easily seen to hold.

It can be shown that for $x > 1$,

$$[(A_pC_p - B_p^2)/(A_0C_0 - B_0^2)]^{1/2} < n^{2p}$$

and

$$(A_p/A_0) \geqslant (n/2)^{2p}\,[x^n/(x^n + 1)].$$

If $K(p)$ is the integrand of (3.1), for all

$$x \geqslant 1 + \delta,\ \delta = \exp[-(\log n)^{1/3}]$$

we get

$$K(p) < 4^{p+1}\, K(0) < \alpha(p)\,(x^2 - 1)^{-1}$$

where $\alpha(p)$ is a constant depending only on p. Hence we get

$$E[N_n(1 + \delta, \infty)] = O(\log \delta).$$

Always

$$(C_p/A_p)^{1/2} < (4p + 2)/x(1 - x^2).$$

If $0 < x < 1/2$, then

$$(C_p/A_p) < \sum_{m=1}^{\infty} m^{2p+2}\, 4^{-m}$$

and hence (3.1) is

$$E[N_n(0, 1 - \delta)] = O(\log \delta).$$

Since $xK(p)$ is less than n we get

$$E[N_n(1 - T/n, 1 + T/n)] = O(T)$$

for $T = (\log n)^{1/2}$. Hence we get (iii) of section 1.

§4. We put

$$U(t) = U(t : \alpha, \beta) = \begin{cases} 1 \text{ if } P(\alpha, t)\, P(\beta, t) < 0 \\ \frac{1}{2} \text{ if } P(\alpha, t)\, P(\beta, t) = 0 \\ 0 \text{ if } P(\alpha, t)\, P(\beta, t) > 0. \end{cases}$$

Let $N(t) = N_n(\alpha, \beta)$ be the number of real zeros of $P(x, t)$ in $\alpha \leqslant x \leqslant \beta$, counted according to their multiplicities except for zeros at α and β, which are counted according to half their multiplicities. From Das (1972, Lemma 2) we get

$$0 \leqslant \int_0^1 [N(t) - U(t)]\, dt \leqslant C\, \gamma^2(-\log \gamma)^{1/2}$$

where $\gamma = n(\beta - \alpha)$ and C is an absolute constant. Let

$$\triangle = n^{-2} \exp [(\log n)^{1/3}]$$

and let q_0, q_1 be such that

$$q_0 \triangle \leqslant (\log n)^{1/2}/n \qquad < (q_0 + 1) \triangle$$

$$q_1 \triangle \leqslant \exp [-(\log n)^{1/3}] \quad < (q_1 + 1) \triangle.$$

If α_q and β_q are defined as

$$\alpha_q = 1 + q \triangle \qquad (q_0 \leqslant q \leqslant q_1)$$

and $$\beta_q = 1 + (q+1) \triangle \quad (q_0 \leqslant q \leqslant q_1)$$

and $N_q(t)$, $U_q(t)$, γ_q are functions corresponding to $N(t)$, $U(t)$, γ for the interval $\alpha_q \leqslant x \leqslant \beta_q$, we have

$$\gamma_q = n(\beta_q - \alpha_q) = n \triangle$$
$$= n^{-1} \exp[(\log n)^{1/3}]$$

and

$$\int_0^1 [N_q(t) - U_q(t)]\, dt \leqslant C\, \gamma_q^2(-\log \gamma_q)^{1/2}$$
$$< C[\exp 2(\log n)^{1/3}]\, (\log n)^{1/2}\, n^{-2}.$$

Thus

$$\int_0^1 \sum_{q=q_0}^{q_1} [N_q(t) - U_q(t)]\, dt$$
$$< C(q_1 - q_0)(\log n)^{1/2}\, n^{-2} \exp[2(\log n)^{1/3}]$$
$$< A(\log n)^{1/2},$$

where A is an absolute constant. Therefore

$$\int_0^1 n(t : 1 + T/n, 1 + \delta)\, dt = \int_0^1 \sum_{q=q_0}^{q_1} N_q(t)\, dt$$

$$= \sum_{q=q_0}^{q_1} \int_0^1 U_q(t)\, dt + O(\log n)^{1/2}.$$

For $x \geqslant 1 + \delta$ we know that $A_p(x) \sim n^{2p+2} x^{2n}/(x^2 - 1)$. Also the function $U(t)$ is integrable and

$$\int_0^1 U_q(t)dt = \pi^{-1} \sin^{-1} \tau_q,$$

where

$$\tau_q^2 = 1 - \frac{\left(\sum_{k=1}^{n} k^{2p+2} \alpha_q^{k-1} \beta_q^{k-1}\right)^2}{\left(\sum_{k=1}^{n} k^{2p+2} \alpha_q^{2k-2}\right)\left(\sum_{k=1}^{n} k^{2p+2} \beta_q^{2k-2}\right)}$$

$$\sim [(\beta_q - \alpha_q)/(\beta_q\alpha_q - 1)]^2.$$

Thus we find

$$\int_0^1 U_q(t)\, dt = (2\pi q)^{-1} + O(\triangle)$$

and hence

$$\int_0^1 n(t : 1 + T/n, 1 + \delta)\, dt = (2\pi)^{-1} \log n + O(\log n)^{1/2}.$$

Using (2.2) we get

$$M_n(1 + T/n, 1 + \delta) = (4\pi)^{-1} \log n + O(\log n)^{1/2}.$$

Hence we get (ii) of section 1.

§5. Here we estimate the value of $M_n(1 - \delta, 1 - T/n)$.

Like Das (1972, section 4) for the range $(1 - \delta, 1 - T/n)$ we follow the method given by Logan and Shepp (1968). We know that the Kac-Rice formula is

$$E[N_n(a, b)] = \lim_{\varepsilon \to 0} (2\epsilon)^{-1} \int_a^b E[\psi_\varepsilon(P(x, t)) \mid P'(x, t) \mid]\, dx,$$

where

$$\psi_\epsilon(x) = \begin{cases} 1 \text{ if } -\epsilon < x < \epsilon \\ 0 \text{ otherwise.} \end{cases}$$

Now the combined variable $(P(x, t), P'(x, t))$ has the characteristic function

$$\phi(z, w) = E[\exp(iP(x, t) z + iP'(x, t)w)].$$

The density function $p(\xi, \eta)$ for $P(x, t) = \xi$ and $P'(x, t) = \eta$ is

$$p(\xi, \eta) = (2\pi)^{-2} \int\int_{-\infty}^{\infty} \exp(-i\xi z - i\eta w)\, \phi(z, w)\, dz\, dw.$$

Taking $P(x, t) = \sum_{k=1}^{n} c_k(t)\, a_k$ and $P'(x, t) = \sum_{k=1}^{n} c_k(t)\, b_k$

we get

$$\phi(z, w) = \exp[-(\tfrac{1}{2}) \sum_{k=1}^{n} (a_k z + b_k w)^2].$$

Since

$$E[N_n(a, b)] = \int_a^b dx \int_{-\infty}^{\infty} |\eta|\, p(0, \eta)\, d\eta$$

and choosing

$$A^2 = \sum_{k=1}^{n} a_k^2 \text{ and } AB = \sum_{k=1}^{n} a_k b_k \text{ we obtain}$$

$$\int_{-\infty}^{\infty} |\eta|\, p(0, \eta)\, d\eta = \pi^{-1} \int_0^{\infty} w^{-2} dw \int_{-\infty}^{\infty} \{\exp[-(\tfrac{1}{2})(Az + Bw)^2]$$

$$- \exp[-(\tfrac{1}{2}) \sum_{k=1}^{n} (a_k z + b_k w)^2]\}\, dz.$$

We put $z = xuw'$, $w = -x^2 w'$. Now changing w' into w and using Frullanies theorem to integrate on w we get

$$\int_{-\infty}^{\infty} |\eta|\, p(0, \eta)\, d\eta = (2x\pi^2)^{-1} \int_{-\infty}^{\infty} \log g_n(u, x)\, du$$

where

$$g_n(u, x) = \sum_{k=1}^{n} (u - (k-1))\, \phi_k(x^2)/(u - \phi(x^2))^2$$

and

$$\phi_k(x) = (k^{2p+2} x^{k-1})/(\sum_{k=1}^{n} k^{2p+2} x^{k-1}),$$

$$\phi(x) = \sum_{k=1}^{n} (k-1)\, \phi_k(x).$$

Putting $a = 1 - \delta$, $b = 1 - T/n$, $x = \exp(-r/2n)$, $u = nv/r$, we get

$$E[N_n(1-\delta, 1-T/n)] = (2\pi)^{-2} \int_{T_0}^{n\delta_0} r^{-1}\,dr \int_{-\infty}^{\infty} \log V(v, r)\,dv \qquad \ldots(5.1)$$

where

$$T_0 = -2n \log(1 - T/n),$$

$$\delta_0 = -2 \log(1-\delta),$$

$$V(v, r) = [v^2 - 2\lambda(r)\,v + \Lambda(r)][v^2 - 2\lambda(r)v + \lambda^2(r)]^{-1},$$

$$\lambda(r) = (r/n)\,\frac{[\sum_{k=1}^{n} k^{2p+2}(k-1)\exp(-kr/n)]}{[\sum_{k=1}^{n} k^{2p+2}\exp(-kr/n)]}$$

and

$$\Lambda(r) = \left(\frac{r}{n}\right)^2 \frac{[\sum_{k=1}^{n} k^{2p+2}(k-1)^2\exp(-kr/n)]}{[\sum_{k=1}^{n} k^{2p+2}\exp(-kr/n)]}.$$

Following the method of Das (1972, section 4) we get

$$(r/n)\,[\sum_{k=1}^{n} k^{2p+3}\exp(-kr/n)]\,[\sum_{k=1}^{n} k^{2p+2}\exp(-kr/n)]^{-1}$$
$$= (2p+3) + O(re^{-r/2})$$

and

$$(r/n)^2\,[\sum_{k=1}^{n} k^{2p+4}\exp(-kr/n)]\,[\sum_{k=1}^{n} k^{2p+2}\exp(-kr/n)]^{-1}$$
$$= (2p+4)(2p+3) + O(re^{-r/2}),$$

for large r. Hence

$$\lambda(r) \sim (2p+2)$$

and $$\Lambda(r) \sim (4p^2 + 10p + 7).$$

Since

$$\pi^{-2} \int_{T_0}^{n\delta_0} r^{-1}\,dr \int_{|v|>L} \log \frac{v^2 - 2\lambda(r)\,v + \Lambda(r)}{v^2 - 2\lambda(r)\,v + \lambda^2(r)}\,dv$$
$$= O(\log n/L)$$

and

$$(2\pi)^{-2} \int_{T_0}^{n\delta_0} r^{-1}\, dr \int_{-L}^{L} \log \frac{v^2 - 2\lambda(r)\, v + \Lambda(r)}{v^2 - 2\lambda(r)\, v + \lambda^2(r)}\, dv$$

$$= (2\pi)^{-2} \int_{T_0}^{n\delta_0} r^{-1}\, dr \int_{-L}^{L} \log \frac{v^2 - 2(2p+2)\, v + 4p^2 + 10p + 7}{v^2 - 2(2p+2)\, v + (2p+2)^2}\, dv + \eta$$

where $\eta = \epsilon \log n$ and ϵ is small, taking L large from (5.1) we get

$$E[N_n(1-\delta,\ 1 - T/n)] = C \log n + O(\log n),$$

where

$$C = (2\pi)^{-2} \int_{-\infty}^{\infty} \log \frac{v^2 - 2v(2p+2) + (4p^2 + 10p + 7)}{v^2 - 2v(2p+2) + (2p+2)^2}\, dv$$

$$= (2\pi)^{-2} \int_{-\infty}^{\infty} \log \left[1 + \frac{(2p+3)}{(v - 2p - 2)^2}\right] dv$$

$$= (2\pi)^{-2} \int_{-\infty}^{\infty} \log \left[1 + \frac{(2p+3)}{v^2}\right] dv$$

$$= (2\pi)^{-1} [(2p+3)]^{1/2}.$$

Hence from (2.2) we get

$$M_n(1-\delta,\ 1 - T/n) = (4\pi)^{-1} [(2p+3)]^{1/2} \log n + O(\log n).$$

Combining the results in sections 3, 4 and 5 we get

$$M_n(-\infty, \infty) = (2\pi)^{-1} [1 + \sqrt{(2p+3)}\,] \log n + O(\log n)^{1/2}.$$

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