A FINITE TRANSFORM INVOLVING GENERALIZED PROLATE SPHEROIDAL WAVE FUNCTIONS AND ITS APPLICATIONS

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In this paper we have developed firstly the finite transform involving generalized prolate spheroidal functions and discussed their properties in detail. Legendre's transform, Gagenbaur transform etc., are its particular cases. Later on it has been applied to solve the physical problems.

INTRODUCTION

In a recent paper Gupta (1968) has investigated a finite transform involving spheroidal wave functions and discussed its properties together with its application in solving the diffusion equations. Now recently generalized prolate spheroidal wave functions have been investigated by Slepian (1965). Hitherto, however, corresponding finite transform involving generalized prolate spheroidal wave functions have not been investigated. The aim of this paper is to supply this deficiency. The corresponding transforms available in the literature (Gupta 1968, Sneddon 1951, Gupta and Gupta 1975) are the particular cases of the transform to be discussed.

In this paper transform applicable to generalized spheroidal wave functions analogous to finite Fourier's transform and Hankel transform (Sneddon 1951) has been investigated together with its properties. Its application to the solution of a few boundary value problems relating to prolate spheroids have been discussed. As regards the spheroidal wave functions the notation used is as given by Slepian (1964).

Theorem I—If any function of $f(\alpha, \beta)$ is continuous and single valued within the spheroid and vanishes on the boundary, then its finite transform analogous to the finite transform Gupta (1968) in the range $0 \leqslant \beta \leqslant 1$, $1 \leqslant \alpha \leqslant \alpha_0$ is defined as

$$K_{N,\,n}(C_{N,\,p}) = \int_{1}^{\alpha_0} \int_{-1}^{1} (\alpha^2 - \beta^2)\, K_{N,\,n}(\alpha,\beta)^*\, \phi_{N,\,n}(\beta)\, \Psi'^{(1)}_{N,\,n}(\alpha)\, d\alpha\, d\beta \quad \ldots(1.1)$$

* $K_{N,\,n}(\alpha, \beta)$ is the solution of the following differential equation :

$$\left(\frac{\partial}{\partial\alpha}(\alpha^2 - 1)\frac{\partial}{\partial\alpha} + \frac{N^2 - 1/4}{\alpha^2} + \frac{\partial}{\partial\beta}(1 - \beta^2)\frac{\partial}{\partial\beta} + \frac{1/4 - N^2}{\beta^2} + c^2(\alpha^2 - \beta^2)\right) \times K_{N,\,n}(\alpha, \beta) = 0$$

where $C_{N,\,p}$ is the pth root of

$$\Psi'^{(1)}_{N,\,n}(\alpha_0) = 0 \qquad \ldots(1.2)$$

and following Slepian (1964)

$$\Psi'^{(1)}_{N,\,n}(\alpha)\ \frac{1}{\gamma_{N,\,n}} \sum_{j=0}^{\infty} d_j^{N,\,n}(C)\ \frac{J_{N+2n+1}(C\alpha)}{(C\alpha)^{1/2}} \binom{N+n}{n}^{-1},\ 0 \leqslant \alpha \leqslant \infty \qquad \ldots(1.3)$$

$$\gamma_{N,\,n} = \frac{C^{N+1/2}\ d_0^{N,\,n}(C)}{2^{N+1}\sqrt{N+2}\ \sum_{j=0}^{\infty} d_j^{N,\,n}(C)} \qquad \ldots(1.4)$$

and $$\phi_{N,\,n}(\beta) = \sum_{0}^{\infty} d_j^{N,\,n}(C)\ T_{N,\,j}(\beta)$$

where $T_{N,\,n}(\beta)$ are closely related to the Zermike polynomials which arise in the diffraction theory of aberrations, or following Slepian we have

$$T_{N,\,n}(\beta)\ \beta^{N+1/2}\ F(-n, n+N+1;\ N+1;\ \beta^2) = \beta^{N+1/2}\ R_{N,\,n}(\beta)$$

where $F(a, b;\ c, z)$ is the usual hypergeometric function. The polynomial $R_{N,\,n}(\beta)$ is readily expressed in terms of Jacobi polynomials $P_n^{\delta,\,\delta'}(\beta)$. Adopting the notation of Seago, we have

$$R_{N,\,n}(\beta) = \binom{n+N}{n}^{-1}\ P_n^{N,\,0}\ (1 - 2\,\beta^2)$$

then at any point within the range

$$K_{N,\,n}(\alpha, \beta) = \sum_{n=0}^{\infty} B_n\ \Psi'^{(1)}_{N,\,n}(\alpha)\ \phi_{N,\,n}(\beta) = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} B_n\ \Psi'^{(1)}_{N,\,n}(\alpha, C_{n,\,p}) \times \phi_{N,\,n}(\beta, C_{n,p}) \qquad \ldots(1.5)$$

where the constant coefficients are to be determined by the following method.

Multiplying equation (1.5) on both the sides by $(\alpha^2 - \beta^2)\,\phi_{N,\,n}(C_{N,\,p}, \beta)\ \Psi'_{N,\,n} \times (C_{N,\,p}, \alpha)$ and integrating w. r. t. β between $0 \leqslant \beta \leqslant 1$ and w. r. t. α between $1 \leqslant \alpha \leqslant \alpha_0$ we have

$$B_n = \frac{\int_1^{\alpha_0} \int_0^1 K_{N,n}(\alpha, \beta)(\alpha^2 - \beta^2) \Psi^{(1)}_{N,n}(C_{N,p}, \alpha)\, \phi_{N,n}(C_{N,p}, \beta)\, d\alpha\, d\beta}{\Lambda_{N,n} \int_1^{\alpha_0} \left[\Psi^{(1)}_{N,n}(C_{N,p}, \alpha) \right]^2 \left[(\alpha^2 - \Psi_{N,p}) \right] d\alpha} \quad \ldots(1.6)$$

$$\Theta_{N,p}(C_{n,p}) = \frac{1}{\Lambda_{N,n}} \int_0^1 \beta^2 \left[\phi_{N,n}(C_{N,p}, \beta) \right]^2 d\beta \quad \ldots(1.7)$$

Following the property Slepian (1964)

$$\beta^2 T_{N,n}(\beta) = \gamma^1_{N,n} T_{N,n+1}(\beta) + \gamma^0_{N,n} T_{N,n}(\beta) + \gamma^{-1}_{N,n} T_{N,n-1}(\beta)$$

where

$$\gamma'_{N,n} = -\frac{(n+N+1)^2}{(2n+N+1)(2n+N+2)}$$

$$\gamma^0_{N,n} = \tfrac{1}{2}\left(1 + \frac{N^2}{(2n+N)(2n+N+2)} \right)$$

$$\gamma^{-1}_{N,n} = -\frac{n^2}{(2n+N)(2n+N+1)}$$

$$\| T_{N,n}(\beta) \| \leqslant | \quad \text{for } 0 \leqslant \beta \leqslant 1.$$

Integral (1.7) can be evaluated and we have

$$\Theta(C_{N,p}) = \frac{1}{\Lambda_{\ ,n}} \left[\sum_{j=0}^{\infty} \left\{ d^{N,n}_j \; d^{N,n}_{j+1} \; \gamma^1_{n,j} \frac{1}{(2j+N+3)\binom{N+j+1}{j+1}} + (d^{N,n}_j)^2 \gamma^0_{N,j} \frac{1}{2(2j+N+1)\binom{N+j}{j}} + d^{N,n}_{j-1} \; d^{N,n}_j \; \gamma^{-1}_{N,j} \frac{1}{2(2j+N-1)\binom{N+j-1}{j-1}} \right\} \right] \quad \ldots(1.8)$$

provided $j - 1 \geqslant 0$

$$\Lambda_{N,n} = \int_0^1 \left[\phi_{N,n}(C_{N,p}\, \beta) \right]^2 = \sum_{j=0}^{\infty} \left[d^{N,n}_j \right]^2 \frac{1}{2(2j+N+1)\binom{n+N}{n}}. \quad \ldots(1.9)$$

Hence, the inversion formula is given by

$$K_{N,n}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} \frac{K_{N,n}(C_{N,p})\, \phi_{N,n}(C_{N,p}, \beta)\, \Psi'^{(1)}_{N,n}(C_{N,p}, \alpha)}{\Lambda_{N,n} \int_{0}^{\alpha_0} [\Psi'_{N,n}(C_{N,p}, \alpha)]^2 [\alpha^2 - \Theta_{N,p}]\, d\alpha}. \quad \ldots(1.10)$$

Definition II—If $K_{N,n}(\alpha, \beta)$ is continuous and single valued in the region $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, $0 \leqslant \beta \leqslant 1$ and vanishes on $\alpha = \alpha_1$, then its finite transform in the range is given by

$$\bar{K}_{N,n}(C_{N,p,n}) = \int_{\alpha_0}^{\alpha_1}\int_{0}^{1} K_{N,n}(\alpha, \beta)(\alpha^2 - \beta^2) B_{N,n}(\alpha)\, \phi_{N,n}(C, \beta)\, d\alpha\, d\beta \quad \ldots(1.11)$$

where

$$B_{N,n}(\alpha) = \Psi'_{N,n}(C_{N,n,p}, \alpha)\, \Psi'^{(2)'}_{N,n}(C_{N,n,p}, \alpha_0) - \Psi'^{(1)'}_{N,n}(C_{N,n,p}, \alpha_0)$$
$$\times\, \Psi'^{(2)}_{N,n}(C_{N,n,p}, \alpha) \quad \ldots(1.12)$$

and prime denotes differentiation with respect to α, then put $\alpha = \alpha_0$ whereas $C_{N,n,p}$ is the pth root of the equation

$$\Psi'_{N,n}(C_{N,n,p}, \alpha_1)\, \Psi'^{(2)'}_{N,n}(C_{N,n,p}, \alpha_0)$$
$$- \Psi'^{(1)'}_{N,n}(C_{N,n,p}, \alpha_0)\, \Psi'^{(2)}_{N,n}(C_{N,n,p}, \alpha_0) = 0 \quad \ldots(1.13)$$

and

$$\Psi'^{(2)}_{N,n}(C_{N,n,p}, \alpha) = \frac{1}{\gamma(N, n)} \sum_{j=0}^{\infty} d_j^{N,n}(C_{N,n,p})$$
$$\frac{J_{-N-2n-1}(C_{N,n,p}, \alpha)}{(C_{N,n,p}, \alpha)^{\frac{1}{2}}} \binom{N+n}{n}^{-1}, \; 1 \leqslant \alpha < \infty. \quad \ldots(1.14)$$

Obviously, the function $K_{N,p}(\alpha, \beta)$ by the well known generalised Fourier's series within the region can be expressed as

$$K_{N,n}(\alpha, \beta) = \sum_{n=0}^{\infty} D_n B_{N,n}(\alpha)\, \phi_{N,n}(\beta)$$

$$= \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} D_n B_{N,n}(C_{N,n,p}, \alpha)\, \phi_{N,n}(C_{N,n,p}, \beta).$$

The constant D_n can be determined by the same method as in Theorem I, viz.

$$D_n = -\frac{\int_{\alpha_0}^{\alpha_1}\int_0^1 K_{N,\,n}(\alpha,\,\beta)\,(\beta^2-\alpha^2)\,B_{N,\,n}(C_{N,\,n,\,p},\,\alpha)\,\phi_{N,\,n}(C_{N,\,n,\,p},\,\beta)\,d\alpha\,d\beta}{\Lambda_{N,\,n}\int_{\alpha_0}^{\alpha_1}[B_{N,\,n}(C_{n,\,p,\,N},\,\alpha)]^2\,[\alpha^2-\Theta_{N,\,n}]\,d\alpha} \qquad \ldots(1.15)$$

where $\Lambda_{N,\,n}$ is of the same form as in (1.8) but different argument in $d_j^{N,\,n}$'s.

Hence, the inversion formula is given by

$$\bar{K}_{N,\,n}(\alpha,\,\beta) = \sum_{n=0}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{K}_{N,\,n}(C_{N,\,n,\,p})\,B_{N,\,n}(C_{n,\,N,\,p},\,\alpha)\,\phi_{N,\,n}(C_{N,\,n,\,p},\,\beta)}{\Lambda_{N,\,n}\int_{\alpha_0}^{\alpha_1}[B_{N,\,n}(C_{N,\,n,\,p},\,\alpha)^2][\alpha^2-\Theta_{N,\,p}]\,d\alpha} \qquad \ldots(1.16)$$

Definition III—If any function of $(\alpha,\,\beta)$ is continuous and single valued in the region $\alpha_0 \leqslant \alpha < \alpha_1$, $0 \leqslant \beta \leqslant 1$, vanishing on both the boundaries $\alpha = \alpha_0$, and $\alpha = \alpha_1$ then its finite transform in the region is defined as

$$\bar{K}_{N,\,n}(C_{N,\,n,\,p}) = \int_{\alpha_0}^{\alpha_1}\int_0^1 K_{N,\,n}(\alpha,\,\beta)\,(\alpha^2-\beta^2)\,B_{N,\,n}(\alpha_1)\,\phi_{N,\,n}(C_{n,\,N,\,p},\,\beta)\,d\alpha\,d\beta \qquad \ldots(1.17)$$

where

$$B_{N,\,n}(C_{N,\,n},\,\alpha) = \Psi'^{(2)}_{N,\,n}(C_{N,\,n,\,p},\,\alpha_0)\,\Psi'^{(1)}_{N,\,n}(C_{N,\,n,\,p},\,\alpha)$$
$$-\,\Psi'^{(2)}_{N,\,n}(C_{N,\,n\,p},\,\alpha)\,\Psi'^{(1)}_{N,\,n}(C_{N,\,n,\,p},\,\alpha_0) \qquad \ldots(1.18)$$

and

$C_{N,\,n,\,p}$ is the pth root of the equation

$$\Psi'^{(2)}_{N,\,n}(C_{N,\,n,\,p},\alpha_0)\,\Psi'^{(1)}_{N,\,n}(C_{N,\,n,\,p},\,\alpha_1)$$
$$-\,\Psi'^{(2)}_{N,\,n}(C_{N,\,n},\,\alpha_1)\,\Psi'^{(1)}_{N,\,n}(C_{N,\,n,\,p},\,\alpha_0) = 0. \qquad \ldots(1.19)$$

Adopting the same method as in Theorem I and Definition II above we have

$$K_{N,\,n}(\alpha,\,\beta) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} D_n\,B_{N,\,n}(C_{N,\,p,\,n},\,\alpha)\,\phi_{N,\,n}(C_{N,\,n,\,p},\,\beta),$$

where

$$D_N = \frac{\bar{K}_{N,\,n}(C_{N,\,n,\,p})}{\Lambda_{N,\,n} \int_{\alpha_0}^{\alpha_1} [B_{N,\,n}(C_{N,\,n,\,p},\,\alpha)]^2\,[\alpha^2 - \Theta_{N,\,n,\,p}]\,d\alpha}$$

Hence the inversion formula is expressed as

$$K_{N,\,n}(\alpha,\,\beta) = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\bar{K}_{N,\,n}(C_{N,\,n,\,p})\,B_{N,\,n}(C_{N,\,n,\,p},\,\alpha)\,\phi_{N,\,n}(C_{N,\,n,\,p},\,\beta)}{N_{N,\,n} \int_{\alpha_0}^{\alpha_1} [B_{N,\,n}(C_{N,\,n,\,p},\,\alpha)]^2\,[\alpha^2 - \Theta_{N,\,n,\,p}]\,d\alpha} \qquad \ldots(1.20)$$

where $\Lambda_{N,\,n}$ and $C_{N,\,n,\,p}$ are of the same form except the change in the argument of $d_j^{N,\,n}$'s which is the zero of (1.19).

2. Properties of the Finite Transform Defined Above

In the application of the theory of the transform discussed above, we shall require some simple properties of this transform for the solution of special problems. Hence, in the following section we shall discuss them.

Case I—Let $C_{N,\,n,\,p}$ be the root of eqn. (1.2) and

(i) $$\left[(1 - \beta^2) \left\{ \phi_{N,\,n}(\beta) \frac{\partial K_{N,\,n}}{\partial \beta} - K_{N,\,n} \frac{\partial \phi_{N,\,n}}{\partial \beta} \right\} \right]_0^1 = 0$$

(ii) Let $K_{N,\,n}(\alpha,\,\beta)$ along with its first and second derivatives be continuous functions of α and β. Also let $K_{N,\,n}(\alpha,\,\beta)$ satisfy the equation

$$\left[L_\alpha + L_\beta + C_N^2\,(\alpha^2 - \beta^2) \right] K_{N,\,n}(\alpha,\,\beta) = 0 \qquad \ldots(2.1)$$

in the domain of the α plane which includes the interval $1 \leqslant \alpha \leqslant \alpha_0$ and in the domain of β-plane that includes the interval $0 \leqslant \beta \leqslant 1$ where

$$\left.\begin{aligned} L_\alpha &= \frac{\partial}{\partial \alpha}(\alpha^2 - 1)\,\frac{\partial}{\partial \alpha} + \frac{N^2 - \frac{1}{4}}{\alpha^2} \\ L_\beta &= \frac{\partial}{\partial \beta}(1 - \beta^2)\,\frac{\partial}{\partial \beta} + (\tfrac{1}{4} - N^2)/\beta^2 \end{aligned}\right\} \qquad \ldots(2.1a)$$

Now

$$\int_1^{\alpha_0} \int_0^1 \phi_{N,\,n}(\beta)\,\Psi_{N,\,n}(\alpha)\,[\,L_\alpha + L_\beta\,]\,K_{N,\,n}(\alpha,\,\beta)\,d\alpha\,d\beta = I_1 + I_2 \qquad \ldots(2.2)$$

where

$$I_1 = \int_1^{\alpha_0} \int_0^1 \phi_{N,\,n}\,\Psi_{N,\,n}\,L_\alpha\,K_{N,\,n}\,(\alpha,\,\beta)\,d\alpha\,d\beta \qquad \ldots(2.3)$$

$$I_2 = \int_1^{\alpha_0} \int_0^1 \phi_{N,\,n}\,\Psi_{N,\,n}\,L_\beta\,K_{N,\,n}(\alpha,\,\beta)\,d\alpha\,d\beta. \qquad \ldots(2.4)$$

Integrating (2.2) by parts we get

$$I_1 = \int_0^1 \phi_{N,\,n}(\beta)\left[(\alpha^2 - 1)\left\{\Psi_{N,\,n}\frac{\partial K_{N,\,n}}{\partial\alpha} - K_{N,\,n}\frac{\partial \Psi_{N,\,n}}{\partial\alpha}\right\}\right]_1^{\alpha_0} d\beta$$

$$+ \int_1^{\alpha_0} \int_0^1 K_{N,\,n}\frac{\partial}{d\alpha}\left\{(\alpha^2 - 1)\frac{\partial \Psi_{N,\,n}}{\partial\alpha}\right\}\phi_{N,\,n}(\beta)\,d\alpha\,d\beta$$

$$+ \int_1^{\alpha_0} \int_0^1 \frac{N^2 - 1/4}{\alpha^2}\,K_{N,\,n}(\alpha,\,\beta)\,\phi_{N,\,n}(\beta)\,\Psi_{N,\,n}(\alpha)\,d\alpha\,d\beta$$

the first integral vanishes at $\alpha = \alpha_0$ and $\alpha = 1$, provided we assume that $K_{N,\,n}(\alpha,\,\beta) = 0$ at $\alpha = \alpha_0$, then

$$I_1 = \int_1^{\alpha_0} \int_0^1 K_{N,\,n}\left[\frac{\partial}{\partial\alpha}(\alpha^2 - 1)\frac{\partial \Psi_{N,\,n}}{\partial\alpha} + \frac{N^2 - 1/4}{\alpha^2}\right] d\alpha\,d\beta.$$

Similarly $$I_2 = \int_1^{\alpha_0}\left[(1 - \beta^2)\left\{\phi_{N,\,n}\frac{\partial K_{N,\,n}}{\partial\beta} - K_{N,\,n}\frac{\partial \phi_{N,\,n}}{\partial\beta}\right\}\right]_0^1 \Psi_{N,\,n}(\alpha)\,d\alpha$$

$$+ \int_0^1 \int_1^{\alpha_0}\left[\frac{\partial}{\partial\beta}(1-\beta^2)\frac{\partial \phi_{N,\,n}}{\partial\beta} - \frac{(N^2-1/4)}{\beta^2}\phi_{N,\,n}\right] K_{N,\,n}\,\Psi_{N,\,n}(\alpha)\,d\alpha\,d\beta.$$

Obviously, the first integral vanishes on both the limits, since

$T_{N,\,n}(1) = -\,T_{N,\,n}(-1)$, hence

$$I_2 = \int_1^{\alpha_0} \int_0^1 K_{N,\,n}\left[\frac{\partial}{\partial\beta}(1-\beta^2)\frac{\partial \phi_{N,\,n}}{\partial\beta} - \frac{N^2 - 1/4}{\alpha^2}\phi_{N,\,n}\right]$$

$$\times\,\Psi_{N,\,n}(\alpha)\,d\alpha\,d\beta.$$

Thus we find that (2.2) transform into

$$I_1 + I_2 = -C^2_{N,n,p} \int_1^{\alpha_0} \int_{-1}^{1} K_{N,n}(\alpha, \beta)(\alpha^2 - \beta^2)\, \phi_{N,n}(C_{N,n,p}, \beta)\, \Psi_{N,n}(\alpha)\, d\alpha\, d\beta$$

$$- \int_0^1 \phi_{N,n}(\beta) \left[(\alpha^2 - 1)\, K_{N,n}(\alpha) \frac{\partial \Psi_{N,n}}{\partial \alpha} \right]_{\alpha=\alpha_0} d\beta$$

by virtue of (2.1). Hence we see that

$$\int_1^{\alpha_0} \int_0^1 \phi_{N,n}(\beta)\, \Psi_{N,n}(\alpha)\, (L_\alpha + L_\beta)\, K_{N,n}(\alpha, \beta)\, d\alpha\, d\beta$$

$$= -C^2_{N,n,p}\, \bar{K}_{N,n,p}(C_{N,n,p}) - \int_0^1 (\alpha_0^2 - 1) \left[K_{N,n} \frac{\partial \Psi_{N,n}}{\partial \alpha} \right]_{\alpha=\alpha_0} \times \phi_{N,n}(\beta)\, d\beta. \qquad \ldots(2.5)$$

Case II — If $C_{N,n,p}$ is a root of equation (1.12) and

(i) $\left[(- \beta^2) \left\{ \phi_{N,n} \frac{\partial K_{N,n}}{\partial \beta} - K_{N,n} \frac{\partial \phi}{\partial \beta} \right\} \right]_0^1 = 0$

(ii) the same as in case I, then

$$\int_{\alpha_0}^{\alpha_1} \int_0^1 \left[(L_\alpha + L_\beta)\, K_{N,n}(\alpha, \beta) \right] B_{N,n}(\alpha)\, \phi_{N,n}(\beta)\, d\alpha\, d\beta \qquad \ldots(2.6)$$

by the same procedure as in case I (2.6) transform into

$$= -(\alpha_0^2 - 1)\, B_{N,n}(C_{N,n,p}, \alpha_0) \int_0^1 \left(\frac{\partial K_{N,n}}{\partial \alpha} \right)_{\alpha=\alpha_0} \phi_{N,n}(C_{N,n,p}, \beta)\, d\beta$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 K_{N,n} \frac{\partial}{\partial \alpha} \left\{ (\alpha^2 - 1) \frac{\partial B_{N,n}}{\partial \alpha} \right\} \phi_{N,n}(\beta)\, d\alpha\, d\beta$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 K_{N,n} \frac{\partial}{\partial \beta} \left\{ (1 - \beta^2) \frac{\partial \phi_{N,n}}{\partial \beta} \right\} B_{N,n}(\alpha)\, d\alpha\, d\beta$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 \left(\frac{N^2 - 1/4}{\alpha^2} - \frac{N^2 - 1/4}{\beta^2} \right) K_{N,n}(\alpha, \beta)\, \phi_{N,n}(\beta)\, B_{N,n}(\alpha)\, d\alpha\, d\beta. \quad \ldots(2.7)$$

Hence (2.6) becomes

$$- (\alpha_0^2 - 1) \int_0^1 \int_{\alpha_0}^{\alpha_1} B_{N,\,n}(C_{N,\,p,\,n},\,\alpha_0) \left(\frac{\partial K_{N,\,n}}{\partial \alpha} \right)_{\alpha=\alpha_0} \phi_{N,\,n}(C_{N,\,n,\,p},\,\beta)\, d\beta$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 K_{N,\,n}(L_\alpha + L_\beta)\, B_{N,\,n}(\alpha)\, \phi_{N,\,n}(\beta)\, d\alpha\, d\beta$$

$$= - (\alpha_0^2 - 1)\, B_{N,\,n}(C_{N,\,n,\,p},\,\alpha_0) \int_{-1}^{1} \left(\frac{\partial K_{N,\,n}}{\partial \alpha} \right)_{\alpha=\alpha_0} \phi_{N,\,n}(\beta)\, d\beta$$

$$- C^2_{N,\,n,\,p}\, \overline{K}_{N,\,n}(C_{N,\,n,\,p}) \qquad \ldots(2.8)$$

where $B_{N,\,n}(\alpha)\, \phi_{N,\,n}(\beta)$ is the solution of

$$[L_\alpha + L_\beta + C^2_{N,\,n,\,p}(\alpha^2 - \beta^2)]\, K_{N,\,n}(\alpha,\,\beta) = 0.$$

Case III — If $C_{N,\,n,\,p}$ is a root of eqn. (1.17) and (i) and (ii) are just like the same as before in I and II then proceeding exactly as has been done in deriving equation (2.8) we have

$$\int_{\alpha_0}^{\alpha_1} \int_0^1 B_{N,\,n}(\alpha)\, \phi_{N,\,n}(\beta)(L_\alpha + L_\beta)\, K_{N,\,n}(\alpha,\,\beta)\, d\alpha\, d\beta$$

$$= \int_{\alpha_0}^{\alpha_1} \left[(1 - \beta^2) \left\{ \phi_{N,\,n}(\beta) \frac{\partial K_{N,\,n}}{\partial \beta} - K_{N,\,n} \frac{\partial \phi_{N,\,n}}{\partial \beta} \right\} \right]_0^1 B_{N,\,n}(C_{n,\,N,\,p},\,\alpha)\, d\alpha$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 K_{N,\,n}(\alpha,\,\beta) \left\{ \frac{\partial}{\partial \beta} (1 - \beta^2) \frac{\partial \phi_{N,\,n}}{\partial \beta} \right\} B_{N,\,n}(C_{N,\,n,\,p},\,\alpha)\, d\alpha\, d\beta$$

$$+ \int_0^1 \phi_{N,\,n}(\beta) \left[(\alpha^2 - 1) \left\{ B_{N,\,n}(\alpha) \frac{\partial K_{N,\,n}}{\partial \alpha} - K_{N,\,n} \frac{\partial B_{N,\,n}}{\partial \alpha} \right\} \right]_{\alpha_0}^{\alpha_1} d\beta$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 K_{N,\,n} \left\{ \frac{\partial}{\partial \beta} (\alpha^2 - 1) \frac{\partial \beta_{N,\,n}}{\partial \alpha} \right\} \phi_{N,\,n}(\beta)\, d\alpha\, d\beta$$

$$+ \int_{\alpha_0}^{\alpha_1} \int_0^1 \left(\frac{N^2 - 1/4}{\alpha^2} - \frac{N^2 - 1/4}{\beta^2} \right) K_{N,\,n}(\alpha,\,\beta)\, B_{N,\,n}(\alpha)\, \phi_{N,\,n}(\beta)\, d\alpha\, d\beta$$

$$= \int_{\alpha_0}^{\alpha_1} \int_0^1 K_{N,\,n}(L_\alpha + L_\beta)\, B_{N,\,n}(\alpha)\, \phi_{N,\,n}(\beta)\, d\alpha\, d\beta \qquad \ldots(2.9)$$

where we have assumed that $K_{N,n}$ vanishes on both the boundaries i.e. at $\alpha = \alpha_0$ and $\alpha = \alpha_1$. Hence (2.7) becomes

$$\int_0^1 \int_{\alpha_0}^{\alpha_1} (L_\alpha + L_\beta) K_{N,n} B_{N,n}(\alpha) \phi_{N,n}(\beta)\, d\alpha\, d\beta$$

$$= - C^2_{N,n,p} \int_{\alpha_0}^{\alpha_1} \int_0^1 (\alpha^2 - \beta^2) K_{N,n}(\alpha, \beta) \phi_{N,n}(\beta) B_{N,n}(\alpha)\, d\alpha\, d\beta$$

$$= - C^2_{N,n,p} \bar{K}_{N,n}(C_{N,n,p}) \qquad \ldots(2.10)$$

by virtue of the theorem (Sneddon 1951), where $B_{N,n}(C_{N,n,p}, \alpha)\, \phi_{N,n}(\beta)$ is the solution of

$$[L_\alpha + L_\beta + C^2_{n,p}(\alpha^2 - \beta^2)] K_{N,n}(\alpha, \beta) = 0.$$

It can be readily seen that if $N = -\frac{1}{2}$ these transforms reduced to those of prolate spheroidal wave functions derived by Gupta (1968).

3. Application of the Transform Discussed Above

In this section we have studied the heat transfer from the prolate spheroid which is maintained at a constant temperature and there is a source of heat generation depending upon spheroidal coordinates.

Case I — Here in this case the heat equation is

$$K \nabla^2 \theta + C_0(x, y, z)\, \theta = \rho\, C_p \frac{\partial \theta}{\partial t} + Q,\ 0 \leqslant \beta \leqslant 1,\ 0 \leqslant \alpha \leqslant \alpha_0 \qquad \ldots(3.1)$$

Equation (3.1) transforms into prolate spheroidal coordinates

$$(L_\alpha + L_\beta)\, \theta = kd^2(\alpha^2 - \beta^2) \frac{\partial \theta}{\partial t} + Qd^2(\alpha^2 - \beta^2) \qquad \ldots(3.2)$$

where L_α, L_β have the same meaning as in (2.1a), d is the interfocal distance and $C_0(x, y, z)$ in prolate spheroidal coordinates is given by

$$-(\alpha^2 - \beta^2)(N^2 - 1/4)/\alpha^2\beta^2.$$

The equation (3.2) is to be solved under the boundary conditions.

(i) $\theta(\alpha, \beta, t) = \theta_0$, if $t = +0$, $1 \leqslant \alpha \leqslant \alpha_0$, $-1 \leqslant \beta \leqslant 1$;

(ii) $\theta(\alpha, \beta, t) = 0$, $\alpha = \alpha_0$, $t > 0$, $-1 \leqslant \beta \leqslant 1$. $\qquad \ldots(3.3)$

Let us assume that $C_{N,n,p}$ is a root of the equation

$$\Psi'^{(1)}_{N,n}(C_{N,n,p}, \alpha_0) = 0. \qquad \ldots(3.4)$$

Multiplying equation (3.2) by $\Psi'^{(1)}_{N,n}(C_{N,n,p}, \alpha)\, \phi_{N,n}(C_{N,n,p}, \beta)$ and integrating with respect to α between $0 \leqslant \alpha \leqslant \alpha_0$ and w.r.t. β between $-1 \leqslant \beta \leqslant 1$ we have

$$k\, d^2 \frac{d\bar{\theta}}{dt} C^2_{N,\, n,\, p}\bar{\theta} = -\bar{Q}d^2 \qquad \ldots(3.5)$$

where $\bar{Q} = \int_1^{\alpha_0} \int_{-1}^{1} Q(\alpha^2 - \beta^2)\, \Psi'^{(1)}_{N,\, n}(\alpha)\, \phi_{N,\, n}(\beta)\, d\alpha\, d\beta.$

and
$$\bar{\theta} = \int_1^{\alpha_0} \int_{-1}^{1} \theta(\alpha^2 - \beta^2)\, \Psi'^{(1)}_{N,\, n}(\alpha)\, \phi_{N,\, n}(\beta)\, d\alpha\, d\beta. \qquad \ldots(3.6)$$

The equation (3.6) is to be solved under the boundary conditions

$$\bar{\theta} = 0, \quad t = 0. \qquad \ldots(3.7)$$

Hence the appropriate solution of the problem is

$$\bar{\theta} = \left(\theta_0 + \frac{\bar{Q}d^2}{C^2_{N,\, p,\, n}}\right) e^{-C^2_{N,\, n,\, p}\, t/kd^2} - \frac{\bar{Q}d^2}{C^2_{N,\, p,\, n}} \qquad \ldots(3.8)$$

which by inversion formula we have

$$\theta = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\left[(\theta_0 + \bar{Q}\, d^2/C^2_{N,\, n,\, p}) \exp(-C^2_{N,\, n,\, p} t/kd^2) - \frac{\bar{Q}d^2}{C^2_{N,\, p,\, n}}\right] \phi_{N,\, n}(\beta)\, \Psi_{N,\, n}(\alpha)}{\Lambda_{N,\, n} \int_1^{\alpha_0} \left[\Psi'^{(1)}_{N,\, n}(\alpha)\right]^2 [\alpha^2 - \Theta_{N,\, p}]\, d\alpha} \qquad \ldots(3.9)$$

Case II — Consider the heat transfer in a spheroidal shell such that the outer one is maintained at zero temperature and the inner one is insulated. In that case the problem is

$$(L_\alpha + L_\beta)\, \theta = kd^2(\alpha^2 - \beta^2) \frac{\partial\theta}{\partial t} + Qd(\alpha^2 - \beta^2) \qquad \ldots(3.10)$$

to be solved under the boundary conditions

(i) $\theta(\alpha, \beta, t) = \theta_0$, if $t = 0$ $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, $-1 \leqslant \beta \leqslant 1$;

(ii) $\theta(\alpha, \beta, t) = 0$, $\alpha = \alpha_1$, $t > 0$, $-1 \leqslant \beta \leqslant 1$;

(iii) $\left(\frac{\partial\theta}{\partial\alpha}\right)_{\alpha=\alpha_0} = 0, \quad -1 \leqslant \beta \leqslant 1, \quad t \geqslant 0. \qquad \ldots(3.11)$

Let us assume that $C_{N,\, n,\, p}$ is a root of equation (1.11)

In that case the appropriate solution is

$$\theta = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\left[\left(\theta_0 + \bar{Q}d^2/C^2_{N,\,n,\,p}\right) \exp\left(- C^2_{N,\,n,\,p}\, t/kd^2\right) - \dfrac{\bar{Q}d^2}{C^2_{N,\,n,\,p}}\right)\Big] \times B_{N,\,n}(\alpha, C_{N,\,n,\,p})\, \phi_{N,\,n}(\beta)}{\Lambda_{N,\,n} \int_{\alpha_0}^{\alpha_1} [B_{N,\,n}(\alpha, C_{N,\,n,\,p})]^2\, [\alpha^2 - \Theta_{N,\,n,\,p}]\, d\alpha} \quad \ldots(3.12)$$

where $$\bar{Q} = Q \int_{-1}^{1} \int_{\alpha_0}^{\alpha_1} (\alpha^2 - \beta^2)\, B_{N,\,n}(\alpha, C_{N,\,n,\,p})\, \phi_{N,\,n}(\beta)\, d\alpha\, d\beta. \quad \ldots(3.13)$$

Case III — In case the temperature vanishes on both the boundaries then (3.10) is to be solved under the boundary conditions

(1) $\theta(\alpha, \beta, t) = \theta_0$ if $t = 0,\ \alpha_0 \leqslant \alpha \leqslant \alpha_1,\ -1 \leqslant \beta \leqslant 1$;

(2) $\theta(\alpha, \beta, t) = 0,\quad \alpha = \alpha_0,\ -1 \leqslant \beta, \leqslant 1,\ t > 0$;

(3) $\theta(\alpha, \beta, t) = 0,\quad \alpha = \alpha_1,\ -1 \leqslant \beta \leqslant 1,\ t > 0$. ...(3.14)

Let us assume that $C_{N,\,p,\,n}$ is the pth root of equation (1.17) and hence in that case the appropriate solution is

$$\theta = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\left[\left(\theta_0 + \dfrac{\bar{Q}d^2}{C^2_{N,\,p,\,n}}\right) \exp(-C^2_{N,\,n,\,p}\, t/kd^2) \dfrac{\bar{Q}d^2}{C^2_{N,\,n,\,p}}\right] B_{N,\,n}(\alpha, C_{N,\,n})\, \phi_{N,\,n}(\beta)}{\Lambda_{N,\,n} \int_{\alpha_0}^{\alpha_1} [B_{N,\,n}(\alpha, C_{N,\,n,\,p})]^2\, [\alpha^2 - \Theta_{N,\,n,\,p}]\, d\alpha} \quad \ldots(3.15)$$

where $B_{N,\,n}(\alpha)$ is given by equation (1.16) and

$$\bar{Q} = Q \int_{\alpha_0}^{\alpha_1} \int_{-1}^{1} B_{N,\,n}(C_{N,\,p}, \alpha)\, \phi_{N,\,n}(\beta)(\alpha^2 - \beta^2)\, d\alpha\, d\beta.$$

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