SOME PROPERTIES OF PSEUDO H-PROJECTIVE CURVATURE TENSOR IN A DIFFERENTIABLE MANIFOLD EQUIPPED WITH KH-STRUCTURE

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In this paper we have studied some properties of Pseudo H-projective curvature tensor in a differentiable manifold V_{2m} equipped with KH-structure.

1. INTRODUCTION

We consider a C^{∞} manifold V_{2m}. Let there exist on V_{2m} a vector valued bilinear function F of class C^{∞} such that

$$\bar{\bar{X}} = a^2 X \qquad \text{...(1.1)}$$

for arbitrary vector field X where $F(X) = \bar{X}$ and a is a complex constant. Let us agree to say that F gives to V_{2m} a differentiable structure briefly GF-structure defined by algebraic equation (1.1) (Mishra and Singh 1975a). If the given GF-structure is endowed with a Riemannian metric g such that

$$g(\bar{X}, \bar{Y}) = -\, a^2\, g(X, Y) \qquad \text{...(1.2)}$$

then we say that (F, g) gives to V_{2m} a H-structure.

In the sequel arbitrary vector fields are denoted by $X, Y, Z, \ldots$ etc.

Let us consider a tensor $'F$ of the type (0, 2) in V_{2m} equipped with H-structure such that

$$'F(X, Y) \overset{def}{=} g(\bar{X}, Y) = -\, g(X, \bar{Y}). \qquad \text{...(1.3)}$$

It is easy to calculate

$$'F(\bar{X}, Y) = -\, 'F(X, \bar{Y}) \qquad \text{...(1.4a)}$$

$$'F(\bar{X}, \bar{Y}) = -\, a^2\, 'F(X, Y) \qquad \text{...(1.4b)}$$

$$'F(X, Y) = -\, 'F(Y, X). \qquad \text{...(1.4c)}$$

Let D be the Riemannian connexion with respect to g. If for H-structure

$$(D_X F)(Y) = 0 \qquad \text{...(1.5)}$$

is satisfied, we say that H-structure is KH-structure.

Let R and Ric be the curvature tensor and Ricci tensor of V_{2m}. If V_{2m} is equipped with KH-structure, we have (Yano 1965)

$$R(X, Y, \bar{Z}) = \overline{R(X, Y, Z)} \quad \ldots(1.6a)$$

$$\overline{R(X, Y, \bar{Z})} = a^2 R(X, Y, Z) \quad \ldots(1.6b)$$

$$'R(X, Y, \bar{Z}, \bar{U}) = - a^2 \, 'R(X, Y, Z, U) = \, 'R(\bar{X}, \bar{Y}, Z, U) \quad \ldots(1.6c)$$

where

$$'R(X, Y, Z, U) \overset{def}{=} g(R(X, Y, Z), U)$$

and

$$\text{Ric}\,(\bar{X}, \bar{Y}) = - a^2 \,\text{Ric}\,(X, Y) \quad \ldots(1.6d)$$

$$\text{Ric}\,(\bar{X}, Y) = - \text{Ric}\,(X, \bar{Y}). \quad \ldots(1.6e)$$

The Pseudo H-projective curvature tensor P of V_{2m} equipped with GF-structure is given by (Mishra and Singh 1975b)

$$\begin{aligned} 2a^2P(X, Y, Z) = {} & 2a^2R(X, Y, Z) + a^2Q(Y, Z)\, X - a^2Q(X, Z)\, Y \\ & - a^2[Q(X, Y) - Q(Y, X)]\, Z + Q(Y, \bar{Z})\bar{X} \\ & - Q(X, \bar{Z})\, \bar{Y} - [Q(X, \bar{Y}) - Q(Y, \bar{X})]\, \bar{Z} \end{aligned} \quad \ldots(1.7)$$

where

$$Q(X, Y) = - \frac{2}{2m+2}\left\{\text{Ric}\,(X, Y) + \frac{2}{2m-2}\,(O(\text{Ric}\,(X, Y) + \text{Ric}\,(Y, X)))\right\}.$$

The pseudo H-projective curvature tensor in V_{2m} equipped with KH-structure is given by

$$\begin{aligned} P(X, Y, Z) = {} & R(X, Y, Z) - \frac{1}{2(m+1)a^2}\,[a^2 \,\text{Ric}\,(Y, Z)\, X - a^2 \,\text{Ric}\,(X, Z)\, Y \\ & - \text{Ric}\,(\bar{Y}, Z)\, \bar{X} + \text{Ric}\,(\bar{X}, Z)\bar{Y} - 2\,\text{Ric}\,(X, \bar{Y})\, \bar{Z}] \end{aligned} \quad \ldots(1.8)$$

If we put

$$'P(X, Y, Z, U) \overset{def}{=} g(P(X, Y, Z), U) \quad \ldots(1.9a)$$

we have

$$\begin{aligned} 'P(X, Y, Z, U) = {} & 'R(X, Y, Z, U) - \frac{1}{2(m+1)\, a^2}\,[a^2 \,\text{Ric}\,(Y, Z)\, g(X, U) \\ & - a^2 \,\text{Ric}\,(X, Z)\, g(Y, U) - \text{Ric}\,(\bar{Y}, Z)\, g(\bar{X}, U) \\ & + \text{Ric}\,(\bar{X}, Z)\, g(\bar{Y}, U) - 2\,\text{Ric}\,(X, \bar{Y})\, g(\bar{Z}, U)]. \end{aligned} \quad \ldots(1.9b)$$

We know that the manifold V_{2m} is called a recurrent manifold if

$$(D_U R)(X, Y, Z) = A(U) R(X, Y, Z). \quad ...(1.10)$$

It is called a Ricci-recurrent manifold if

$$(D_U \operatorname{Ric})(Y, Z) = A(U) \operatorname{Ric}(Y, Z). \quad ...(1.11)$$

A manifold V_{2m} equipped with KH-structure will be called pseudo H-projective recurrent if

$$(D_U P)(X, Y, Z) = A(U) P(X, Y, Z). \quad ...(1.12)$$

The manifold V_{2m} equipped with KH-structure will be called symmetric if

$$(D_U R)(X, Y, Z) = 0. \quad ...(1.13)$$

It is called Ricci-symmetric if

$$(D_U \operatorname{Ric})(X, Y) = 0 \quad ...(1.14)$$

and is called pseudo H-projective symmetric if

$$(D_U P)(X, Y, Z) = 0. \quad ...(1.15)$$

Weyle curvature tensor W, conformal curvature tensor V, conharmonic curvature tensor L and concircular curvature tensor C are given by (Sinha 1972)

$$W(X, Y, Z) = R(X, Y, Z) - \frac{1}{2m-1}[X \operatorname{Ric}(Y, Z) - Y \operatorname{Ric}(X, Z)] \quad ...(1.16)$$

$$V(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(m-1)}[\operatorname{Ric}(Y, Z) X - \operatorname{Ric}(X, Z) Y - g(X, Z) R(Y) + g(Y, Z) R(X)] + \frac{r}{2(m-1)(2m-1)} \times [g(Y, Z) X - g(X, Z) Y] \quad ...(1.17)$$

$$L(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(m-1)}[g(Y, Z) R(X) - g(X, Z) R(Y) + \operatorname{Ric}(Y, Z) X - \operatorname{Ric}(X, Z) Y] \quad ...(1.18)$$

$$C(X, Y, Z) = R(X, Y, Z) - \frac{r}{2m(2m-1)}[g(Y, Z) X - g(X, Z) Y] \quad ...(1.19)$$

Eliminating R from (1.16), (1.17), (1.18), (1.19) and (1.8) we get the following expressions with the help of (1.6e)

$$P(X, Y, Z) = W(X, Y, Z) + \frac{3}{2(2m-1)(m+1)}[\operatorname{Ric}(Y, Z) X - \operatorname{Ric}(X, Z) Y] - \frac{1}{2(m+1) a^2}[\operatorname{Ric}(Y, \bar{Z}) \bar{X} - \operatorname{Ric}(X, \bar{Z}) \bar{Y} - 2\operatorname{Ric}(X, \bar{Y}) \bar{Z}] \quad ...(1.20)$$

$$
\begin{aligned}
P(X, Y, Z) = {} & V(X, Y, Z) + \frac{1}{m^2 - 1}[\mathrm{Ric}\,(Y, Z)\, X - \mathrm{Ric}\,(X, Z)\, Y] \\
& + \frac{1}{2(m-1)}[g(Y, Z)\, R(X) - g(X, Z)\, R(Y)] \\
& - \frac{r}{2(m-1)(2m-1)}[g(Y, Z)\, X - g(X, Z)\, Y] \\
& - \frac{1}{2(m+1)a^2}[\mathrm{Ric}\,(Y, \bar{Z})\, \bar{X} - \mathrm{Ric}\,(X, \bar{Z})\, \bar{Y} \\
& - 2\, \mathrm{Ric}\,(X, \bar{Y})\, \bar{Z}] \qquad \ldots(1.21)
\end{aligned}
$$

$$
\begin{aligned}
P(X, Y, Z) = {} & L(X, Y, Z) + \frac{1}{m^2 - 1}[\mathrm{Ric}(Y, Z)\, X - \mathrm{Ric}(X, Z)\, Y] \\
& + \frac{1}{2(m-1)}[g(Y, Z)\, R(X) - g(X, Z)\, R(Y)] \\
& - \frac{1}{2(m+1)\, a^2}[\mathrm{Ric}\,(Y, \bar{Z})\, \bar{X} - \mathrm{Ric}\,(X, \bar{Z})\, \bar{Y} \\
& - 2\, \mathrm{Ric}\,(X, \bar{Y})\, \bar{Z}] \qquad \ldots(1.22)
\end{aligned}
$$

$$
\begin{aligned}
P(X, Y, Z) = {} & C(X, Y, Z) + \frac{r}{2m(2m-1)}[g(Y, Z)\, X - g(X, Z)\, Y] \\
& - \frac{1}{2(m+1)\, a^2}[a^2\, \mathrm{Ric}\,(Y, Z)\, X - a^2\, \mathrm{Ric}\,(X, Z)\, Y \\
& + \mathrm{Ric}\,(Y, \bar{Z})\, \bar{X} - \mathrm{Ric}\,(X, \bar{Z})\, \bar{Y} - 2\, \mathrm{Ric}\,(X, \bar{Y})\bar{Z}]. \qquad \ldots(1.23)
\end{aligned}
$$

2. PSEUDO H-PROJECTIVE CURVATURE TENSOR

Theorem 2.1 — The manifold V_{2m} equipped with KH-structure is pseudo H-projectively flat if and only if it is of constant holomorphic sectional curvature.

PROOF : If the space is pseudo H-projectively flat i.e. $P(X, Y, Z) = 0$, then (1.8) and (1.9b) reduce to

$$
\begin{aligned}
R(X, Y, Z) = {} & \frac{1}{2(m+1)a^2}[a^2\, \mathrm{Ric}\,(Y, Z)\, X - a^2\, \mathrm{Ric}\,(X, Z)\, Y - \mathrm{Ric}\,(\bar{Y}, Z)\, \bar{X} \\
& + \mathrm{Ric}\,(\bar{X}, Z)\, \bar{Y} - 2\, \mathrm{Ric}\,(X, \bar{Y})\, \bar{Z}] \qquad \ldots(2.1)
\end{aligned}
$$

$$
\begin{aligned}
'R(X, Y, Z, U) = {} & \frac{1}{2(m+1)\, a^2}[a^2\, \mathrm{Ric}\,(Y, Z)\, g(X, U) - a^2\, \mathrm{Ric}\,(X, Z)\, g(Y, U) \\
& - \mathrm{Ric}(\bar{Y}, Z)\, g(\bar{X}, U) + \mathrm{Ric}(\bar{X}, Z)\, g(\bar{Y}, U) - 2\, \mathrm{Ric}(X, \bar{Y})\, g(\bar{Z}, U)] \qquad \ldots(2.2)
\end{aligned}
$$

From (2.2), we have

$$\text{Ric}(X, Y) = \frac{r}{2m} g(X, Y). \quad \text{...(2.3)}$$

Hence r is constant. Let us put $r = -\frac{2m(2m+2)}{4} ka^2$, where k is holomorphic sectional curvature, then (2.3) becomes

$$\text{Ric}(X, Y) = -a^2 \frac{2m+2}{4} kg(X, Y). \quad \text{...(2.4)}$$

From (2.2) and (2.4) we get

$$'R(X, Y, Z, U) = -\frac{1}{4} k[a^2 \{g(X, U) g(Y, Z) - g(Y, U) g(X, Z)\} - \{'F(X, U)\, 'F(Y, Z) - 'F(Y, U)\, 'F(X, Z)\} + 2'F(Z, U)\, 'F(X, Y)]. \quad \text{...(2.5)}$$

The equation (2.5) shows that the space is of constant holomorphic sectional curvature.

Conversely, if the space is of constant holomorphic sectional curvature, by virtue of (2.5) the tensor P vanishes identically. Therefore the space is pseudo H-projectively flat.

Hence the theorem is completely proved.

Theorem 2.2 — For V_{2m} equipped with KH-structure, we have

$$'P(\bar{X}, \bar{Y}, Z, U) = 'P(X, Y, \bar{Z}, \bar{U}) = -a^2\, 'P(X, Y, Z, U) \quad \text{...(2.6a)}$$

$$'P(\bar{X}, Y, Z, U) = -\, 'P(X, \bar{Y}, Z, U) \quad \text{...(2.6b)}$$

$$'P(X, Y, \bar{Z}, U) = -\, 'P(X, Y, Z, \bar{U}) \quad \text{...(2.6c)}$$

$$P(X, Y, \bar{Z}) = \overline{P(X, Y, Z)} \quad \text{...(2.6d)}$$

$$\overline{P(X, Y, \bar{Z})} = a^2 P(X, Y, Z). \quad \text{...(2.6e)}$$

PROOF : Barring Z and U in (1.9b) and using (1.2), (1.6c) and (1.6d) we get

$$'P(X, Y, \bar{Z}, \bar{U}) = -a^2\, 'P(X, Y, Z, U).$$

The other results in (2.6*a*) follows similarly. Barring U in the above equation and using (1.1) we get (2.6*c*). Barring Y in the other part of the expression (2.6*a*) we get (2.6*b*) with the help of (1.1). Barring Z and both sides of equation (1.8) separately and using (1.1) and (1.6*d*) we get (2.6*d*). Barring both the sides of the equation (2.6*d*) and using (1.1) we get (2.6*e*).

Theorem 2.3 — For V_{2m} equipped with KH-structure Bianchi's first identity satisfied by pseudo H-projective curvature tensor is

$$P(X, Y, Z) + P(Y, Z, X) + P(Z, X, Y) = 0 \qquad ...(2.7a)$$

$$'P(X, Y, Z, U) + 'P(Y, Z, X, U) + 'P(Z, X, Y, U) = 0. \qquad ...(2.7b)$$

PROOF : Writing two other equations by cyclic permutation of X, Y, Z, in (1.8) and adding these two equations to (1.8) and using (1.6e) we get (2.7a). The equation (2.7b) follows from (2.7a).

Theorem 2.4 — If a differentiable manifold V_{2m} equipped with KH-structure is recurrent manifold with proportionately 1-form A, it is Pseudo H-projective recurrent manifold with proportionality 1-form A.

PROOF : From (1.8) we have,

$$(D_U P)(X, Y, Z) = (D_U R)(X, Y, Z) - \frac{1}{2(m+1)\,a^2}\,[a^2(D_U \text{Ric})(Y, Z)\,X$$
$$- a^2(D_U \text{Ric})(X, Z)\,Y + (D_U \text{Ric})(Y, \bar{Z})\,\bar{X}$$
$$- (D_U \text{Ric})(X, \bar{Z})\,\bar{Y} - 2(D_U \text{Ric})(X, \bar{Y})\,\bar{Z}]. \qquad ...(2.8)$$

Let the manifold be recurrent. Then the above equation with the help of (1.10) and (1.11) implies (1.12).

Theorem 2.5 — If a differentiable manifold V_{2m} equipped with KH-structure is pseudo H-projective recurrent and Ricci-recurrent, it is recurrent. The proportionately 1-form A is same in all the cases.

PROOF : Let the manifold be Pseudo H-projective recurrent and Ricci-recurrent. Then (2.8) with the help of (1.11) and (1.12) implies (1.10).

Theorem 2.6 — For V_{2m} equipped with KH-structure if any two of the following properties hold, the third also holds : (a) It is either projective or conformal or concircular or conharmonic recurrent with proportionately 1-form A, (b) it is pseudo H-projective recurrent with the proportionately 1-form A, (c) it is Ricci recurrent with proportionately 1-form A.

PROOF : From (1.20) we have

$$(D_U P)(X, Y, Z) = (D_U W)(X, Y, Z) + \frac{3}{2(2m-1)(m+1)}\,[(D_U \text{Ric})(Y, Z)\,X$$
$$-(D_U \text{Ric})(X, Z)\,Y] - \frac{1}{2(m+1)\,a^2}\,[(D_U \text{Ric})(Y, \bar{Z})\bar{X}$$
$$-(D_U \text{Ric})(X, \bar{Z})\,\bar{Y} - 2(D_U \text{Ric})(X, \bar{Y})\,\bar{Z}]. \qquad ...(2.9)$$

Let the manifold be projective and pseudo H-projective recurrent with proportionately 1-form A. Then (2.9) with the help of (1.10) and (1.12) gives

$$\frac{3}{2(2m-1)(m+1)}[\{(D_U \text{Ric})(Y, Z) - \text{Ric}(Y, Z)\, A(U)\}\, X - \{(D_U \text{Ric})(X, Z) - A(U)\, \text{Ric}(X, Z)\}\, Y] - \frac{1}{2(m+1)\, a^2}[\{(D_U \text{Ric})(Y, \bar{Z}) - A(U)\, \text{Ric}(Y, \bar{Z})\}\bar{X} - \{(D_U \text{Ric})(X, \bar{Z}) - A(U)\, \text{Ric}(X, \bar{Z})\}\, \bar{Y} - 2\{(D_U \text{Ric})(X, \bar{Y}) - A(U)\, \text{Ric}(X, \bar{Y})\}\, \bar{Z}] = 0$$

Since the above equation holds for arbitrary X, Y, Z, ... etc. we have (1.11). Similarly we can prove that if the manifold be either conformal or concircular or conharmonic recurrent and Pseudo H-projective recurrent with proportionately 1-form A, it is Ricci-recurrent with proportionately 1-form A. The remaining part of the proof follows the pattern of the proof of the Theorems 2.4 and 2.5.

Theorem 2.7 — For V_{2m} equipped with KH-structure if any two of the following properties hold, the third also holds : (a) It is either projective or conformal or concircular or conharmonic symmetric, (b) it is Pseudo H-projective symmetric, (c) it is Ricci symmetric.

The proof follows the pattern of the proof of the Theorm 2.6.

Remark : For $a = -i$ all the theorems above stated are true for Kähler manifold (Mishra 1970).

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