STRONG PSEUDO-CONVEX PROGRAMMING IN BANACH SPACE

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The duality theory for programming problems having strong pseudo-convex objective function and strong pseudo-concave constraints in partially ordered Banach spaces is presented.

1. INTRODUCTION

Ritter (1969–70) has established the converse duality theorem for a class of programming problems with a pseudo-convex objective function and quasi-concave constraints. He has pointed out that the direct duality theorem does not hold for such a class. The purpose of this note is to isolate a sub-class of programming problems for which direct duality theorem does hold, the converse duality being true as a consequence of Ritter's results. Moreover, our results generalize those of Chandra (1972).

Let Y_0, Y_1, ..., Y_m be partially ordered real Banach spaces with partial ordering induced by the closed convex cone $K_j \subset Y_j$ respectively (i.e. $x_j \geqslant y_j$ if and only if $x_j - y_j \in K_j$) and X a real Banach space. Assume that for $j \neq 0$, K_j has interior points and K_0 has the property: $y \in K_0$ and $-y \in K_0$ implies $y=0$. Let L_{ij} denote the set of all bounded linear operators from Y_i into Y_j. An element $A \in L_{ij}$ will be called positive i.e. $A \geqslant 0$ if $A(K_i) \subset K_j$. Let S be a convex subset of X. A Frechet-differentiable mapping $f : S \to Y_j$ will be called strong pseudo-convex if there is a positive real valued function K on $S \times S$ such that for $x_1, x_2 \in S$, the relation $K(x_1, x_2)\,(f(x_1) - f(x_2)) \geqslant f'(x_2)\,(x_1 - x_2)$ holds, $f'(x_2)$ denotes the F-differential of f at x_2. f is strong pseudo-concave iff $-f$ is strong pseudo-convex. Also, let $F : S \to Y_0$, $g_j : X \to Y_j$, $j=1, \ldots, m$, be F-differentiable mappings. Let R denote the set of all points x in S for which the relation $g_j(x) \geqslant 0$ for all $j=1, \ldots, m$ holds. Let Ω denote the set of all points

$(X, T_{10}, T_{20}, \ldots, T_{m0}) \in X \times L_{10} \times \ldots \times L_{m0}$ for which the relations

$$F'(x) - \sum_{j=1}^{m} T_{j0} \circ g_j'(x) = 0$$

and $T_j \geqslant 0 \quad j=1, \ldots, m$

hold. Consider the following pair of problems.

$(P-P)$: Find an $x_0 \in R$ such that

$F(x) \geqslant F(x_0)$ for all $x \in R$.

$(D-P)$: Find an $(x_1, T_1, \ldots, T_m) \in \Omega$ such that

$$F(x_1) - \sum_{j=1}^{m} T_j \bigcirc g_j(x_1) \geqslant F(x) - \sum_{j=1}^{m} T_{j0} \bigcirc g_j(x)$$

for all $(x, T_{10}, \ldots, T_{m0}) \in \Omega$.

We assume that any feasible solution for either problem satisfies the constraint qualification for that problem (Ritter 1969–70.)

2. Duality Theorems

We begin by stating the following lemma which follows easily from the definition of strong pseudo-convexity.

Lemma 2.1 – If F and g_j are strong pseudo-convex functions, then the function

$h = F + \sum_{j=1}^{m} T_{j0} \bigcirc g_j$ is strong pseudo-convex for $T_{j0} \geqslant 0$ for all $j = 1, \ldots, m$.

In what follows, we shall assume that F is strong pseudo-convex and g_j is strong pseudo-concave for each $j = 1, 2, \ldots, m$.

Theorem 2.1 (*Weak Duality*)—If x_0 is a feasible solution of $(P-P)$ and $(x_1, T_1, \ldots, T_m)$ is a feasible solution of $(D-P)$, then

$$F(x_6) \geqslant F(x_1) - \sum_{j=1}^{m} T_j \bigcirc g_j(x_1). \qquad \ldots(2.1)$$

PROOF: Since $(x_1, T_1, \ldots, T_m) \in \Omega$, we have

$$F'(x_1) - \sum_{j=1}^{m} T_j \bigcirc g_j'(x_1) = 0$$

$$\text{i.e.} \left[F'(x_1) - \sum_{i=1}^{m} T_i \bigcirc g_i'(x_1) \right] (x_0 - x_1) = 0.$$

Applying Lemma 2.1, we have

$$F(x_0)-F(x_1)-\sum_{j=1}^{m} T_j \bigcirc g_j(x_0)+\sum_{j=1}^{m} T_j \bigcirc g_j(x_1)\geqslant 0$$

$$i.e.\quad F(x_0)\geqslant F(x_1)-\sum_{j=1}^{m} T_j \bigcirc g_j(x_1)$$

where the last inequality follows from the fact that x is feasible for $(P-P)$ and $T_j\geqslant 0$ for all j.

Remark: It is a simple consequence of the above theorem that if equality holds in (2.1), then x_0 and $(x_1, T_1, \ldots, T_m)$ are optimal solutions of $(P-P)$ and $(D-P)$ respectively.

Theorem 2.2—If x_0 is an optimal solution $(P-P)$ which satisfies the constraint qualification and has the property : either $g_j(x_0)>0$ or $\mathcal{R}(g_j'(x_0))=Y_j$ $j=1, \ldots, m$, then there exist $T_j \in L_{j0}$ such that $(x_0, T_1, \ldots, T_m)$ is an optimal solution of $(D-P)$ and

$$F(x_0)=F(x_0)-\sum_{j=1}^{m} T_j \bigcirc g_j(x_0).$$

PROOF: Since x_0 is an optimal solution of $(P-P)$, therefore by Theorem 2.2 (Ritter 1969-70) there exit $T_j \in L_{j0}$ such that

$$F'(x_0)=\sum_{j=1}^{m} T_j \bigcirc g_j'(x_0) \qquad \ldots(2.2)$$

$$T_j\geqslant 0 \text{ for all } j \qquad \ldots(2.3)$$

and

$$\sum_{j=1}^{m} T_j \bigcirc g_j(x_0)=0. \qquad \ldots(2.4)$$

(2.2) and (2.3) show that $(x_0, T_1, \ldots, T_m)$ is feasible for $(D-P)$ and (2.4) proves that

$$F(x_0)=F(x_0)-\sum_{j=1}^{m} T_j \bigcirc g_j(x_0). \qquad \ldots(2.5)$$

(2.5), together with Theorem 2.1 and the remark following it gives the desired result.

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