FIXED POINT THEOREMS FOR MAPPINGS WHICH ARE NOT NECESSARILY CONTINUOUS

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Some sufficient conditions have been obtained for self-mappings of a complete metric space which ensure a unique fixed point.

§1. Suppose (E, d) is a complete metric space and T is a self-mapping of E. T is said to be a contraction if

(a) $$d(T(x), T(y)) \leqslant \alpha\, d(x, y) \ \forall\ x, y \in E, \quad 0 \leqslant \alpha < 1$$

This condition ensures a unique fixed point for T. Contraction mappings are necessarily continuous. Kannan (1968) took the following condition :

(b) $$d(T(x), T(y)) \leqslant \alpha\{d(x, T(x)) + d(y, T(y))\} \quad \forall\ x, y \in E, \quad 0 < \alpha < \tfrac{1}{2}.$$

This also yields a unique fixed point for T. However, mappings satisfying this condition are not necessarily continuous (Kannan 1969). Condition (b) has been further analysed. (Srivastava and Gupta 1971, Gupta and Srivastava 1972, and Singh 1970).

A generalization of conditions (a) and (b) was obtained independently by Gupta (1972) and Reich (1971). They took the condition

(c) $$d(T(x), T(y)) \leqslant \alpha\, d(x, T(x)) + \beta\, d(y, T(y)) + \gamma\, d(x, y)$$
$$\forall\ x, y \in E;\ \alpha, \beta, \gamma \text{ non-negative and } \alpha + \beta + \gamma < 1.$$

Here, we take a condition which is different from the above conditions. Precisely, we prove the following :

Theorem 1 — Let (E, d) be a complete metric space and let $T : E \to E$ satisfy

(d) $$d(T^{p+1}(x), T^{p+2}(y)) \leqslant \alpha\, d(T^{p}(x), T^{p+1}(x)) + \beta\, d(T^{p+1}(y), T^{p+2}(y)) + \gamma\, d(T^{p}(x), T^{p+1}(y))$$

$\forall\, x, y \in E$, α, β, γ being non-negative and $\alpha + \beta + \gamma < 1$, and p any non-negative integer. Then T has a unique fixed point.

PROOF : We prove the theorem for $p = 0$. The proof in the general case follows on similar lines.

Condition (d) is now

$$d(T(x), T^2(y)) \leqslant \alpha\, d(x, T(x)) + \beta\, d(T(y), T^2(y)) + \gamma\, d(x, T(y)).$$

We define a sequence of elements $\{x_n\}$ in E as follows :

Let $x_0 \in E$ be arbitrary and let

$$x_n = T(x_{n-1}) = T^n(x_0).$$

Now

$$\begin{aligned} d(x_1, x_2) &= d(T(x_0), T^2(x_0)) \\ &\leqslant \alpha\, d(x_0, T(x_0)) + \beta\, d(T(x_0), T^2(x_0)) + \gamma\, d(x_0, T(x_0)) \\ &= \alpha\, d(x_0, x_1) + \beta\, d(x_1, x_2) + \gamma\, d(x_0, x_1) \end{aligned}$$

$$\therefore \quad d(x_1, x_2) \leqslant \left\{\frac{\alpha + \gamma}{1 - \beta}\right\} d(x_0, x_1).$$

Similarly $$d(x_2, x_3) \leqslant \left\{\frac{\alpha + \gamma}{1 - \beta}\right\} d(x_1, x_2)$$

$$\leqslant \left\{\frac{\alpha + \gamma}{1 - \beta}\right\}^2 d(x_0, x_1)$$

and so on.

Since $\frac{\alpha + \gamma}{1 - \beta} < 1$, the sequence $\{x_n\}$ is Cauchy, and E being complete, $\exists\, \xi \in E$ such that

$$\lim_{n \to \infty} x_n = \xi.$$

We shall show that this ξ is the unique fixed point of T.

Let t be any integer. Then

$$\begin{aligned} d(\xi, T(\xi)) &\leqslant d(\xi, x_t) + d(x_t, T(\xi)) \\ &= d(\xi, x_t) + d(T(\xi), T^2(x_{t-2})) \end{aligned}$$

$$d(\xi, T(\xi)) \leqslant d(\xi, x_t) + \alpha\, d(\xi, T(\xi)) + \beta\, d(T(x_{t-2}), T^2(x_{t-2})) + \gamma\, d(\xi, x_{t-1})$$

which gives

$$(1 - \alpha)\, d(\xi, T(\xi)) \leqslant d(\xi, x_t) + \beta\, d(x_{t-1}, x_t) + \gamma\, d(\xi, x_{t-1}).$$

The expression on the right hand side can be made arbitrarily small by choosing t to be sufficiently large. Hence

$$d(\xi, T(\xi)) = 0$$

i.e. $$\xi = T(\xi).$$

Suppose $\eta \in E$ such that $\eta = T(\eta)$. Then

$$d(\xi, \eta) = d(T(\xi), T^2(\eta))$$
$$\leqslant \alpha d(\xi, T(\xi)) + \beta d(T(\eta), T^2(\eta)) + \gamma d(\xi, T(\eta))$$

Hence

$$(1 - \gamma)\, d(\xi, \eta) \leqslant 0$$
$$d(\xi, \eta) = 0 \qquad \text{i.e. } \xi = \eta$$

This proves the theorem.

We illustrate Theorem 1 by the following example :

Example : Let $E = [0, 1]$ with the usual metric and

$T : E \longrightarrow E$ be defined as

$T(x) = 0 \; x \neq \frac{1}{2}$

$T(\frac{1}{2}) = 1.$

T is not continuous, so it does not satisfy (a); taking $x = \frac{1}{2}$ and $y = 0$, we see that T does not satisfy (b) and hence also (c). But it satisfies (d) with $p = 1$.

Recently, Hardy and Rogers (1973) have generalized condition (d) by taking

(e) $$d(T(x), T(y)) \leqslant \alpha\, d(x,T(x)) + \beta\, d(y, T(y)) + \gamma\, d(x, y) + \delta\, d(x, T(y)) + \epsilon\, d(y, T(x))$$

$\forall\, x, y \in E$ where $\alpha, \beta, \gamma, \delta, \epsilon$ are non-negative and $\alpha + \beta + \gamma + \delta + \epsilon < 1$. So we may obtain a generalization of Theorem 1 as follows

Theorem 2 — Let (E, d) be a complete metric space and $T : E \longrightarrow E$ satisfy

$$d(T^{p+1}(x), T^{p+2}(y)) \leqslant \alpha\, d(T^p(x), T^{p+1}(x)), + \beta\, d(T^{p+1}(y)\, T^{p+2}(y)) + \gamma\, d(T^p(x), T^{p+1}(y)) + \delta\, d(T^p(x), T^{p+2}(y)) + \epsilon\, d(T^{p+1}(y), T^{p+1}(x))$$

$\forall\, x, y \in E$ and $\alpha, \beta, \gamma, \delta, \epsilon$ non-negative and $\alpha + \beta + \gamma + \delta + \epsilon < 1$. Then T has a unique fixed point.

The proof is similar to that of Theorem 1.

§2. Srivastava and Gupta (1971) generalized condition (b) by proving the following :

Theorem A — If T_1 and T_2 are two operators each mapping a complete metric space (E, d) to itself and if

$$d(T_1^p(x), T_2^q(y)) \leqslant \alpha\, d(x, T_1^p(x)) + \beta\, d(y, T_2^q(y))$$

$\forall\ x, y \in E$; α, β being non-negative and $\alpha + \beta < 1$ then T_1 and T_2 have a unique common fixed point.

Clearly, condition (c) admits of a similar generalization to give the following :

Theorem 3 — If T_1 and T_2 are two operators each mapping a complete metric space (E, d) to itself and if

$$d(T_1^p(x), T_2^q(y)) \leqslant \alpha\, d(x, T_1^p(x)) + \beta\, d(y, T_2^q(y)) + \gamma\, d(x, y)$$

$\forall\ x, y \in E$, p, q being positive integers and α, β, γ non-negative and $\alpha + \beta + \gamma < 1$, then T_1 and T_2 have a unique common fixed point. We prove the following theorem.

Theorem 4 — If T, T_1 and T_2 are three operators mapping a complete metric space (E, d) to itself and if $\forall\ x, y \in (E, d)$, α, β, γ non-negative and $\alpha + \beta + \gamma < 1$ we have

(i) $$d(T_1^p(x), T_2^q(y)) \leqslant \alpha\, d(T(x), T_1^p(T(x))) + \beta\, d(T(y), T_2^q(T(y))) + \gamma\, d(x, y)$$

(ii) $d(T(x), T(y)) \leqslant d(x, y)$

(iii) $T_1 T(x) = T T_1(x)$

$T_2 T(x) = T T_2(x)$

then there is a unique common fixed point of T, T_1 and T_2.

PROOF : Using conditions (ii) and (iii), condition (i) becomes

$$d(T_1^p(x), T_2^q(y)) \leqslant \alpha\, d(x, T_1^p(x)) + \beta\, d(y, T_2^q(y)) + \gamma\, d(x, y)$$

By Theorem 3, $\exists\ \xi \in E$ which is a unique common fixed point of T_1 and T_2 i.e. $\xi = T_1(\xi) = T_2(\xi)$.

Now

$$\begin{aligned} d(\xi, T(\xi)) &= d(T_1^p(\xi), T_2^q(T(\xi))) \\ &\leqslant \alpha\, d(T(\xi), T_1^p(T(\xi))) + \beta\, d(T^2(\xi), T_2^q(T^2(\xi))) + \gamma\, d(\xi, T(\xi)) \end{aligned}$$

$$(1 - \gamma)\, d(\xi, T(\xi)) \leqslant \alpha\, d(T(\xi), T(\xi)) + \beta\, d(T^2(\xi), T^2(\xi))$$

$$\therefore \quad d(\xi, T(\xi)) = 0 \text{ i.e. } \xi = T(\xi)$$

Hence ξ is the unique common fixed point of T, T_1 and T_2.

Remarks : (i) If we take $T = I$, Theorem 4 reduces to Theorem 3. This shows that T may have more than one fixed point, but there is only one common fixed point for T, T_1 and T_2.

(ii) The second condition means T is non-expansive. This by itself would not ensure a fixed point for T.

(iii) In view of condition (e) we could replace the condition in Theorem 3 by

$$d(T_1^p(x), T_2^q(y)) \leqslant \alpha\, d(x, T_1^p(x)) + \beta\, d(y, T_2^q(y)) + \gamma\, d(x, y) + \delta\, d(x, T_2^q(y)) + \epsilon\, d(y, T_1^p(x))$$

and condition (i) of Theorem 4 by

$$d(T_1^p(x), T_2^q(y)) \leqslant \alpha\, d(T(x), T_1^p(T(x))) + \beta\, d(T(y), T_2^q(T(y))) + \gamma\, d(x, y) + \delta\, d(T(x), T_2^q(T(y))) + \epsilon\, d(T(y), T_1^p(T(x))).$$

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