AN APPLICATION OF THE DIFFERENCE OPERATOR IN THE STUDY OF SOME POLYNOMIALS

by B. M. AGRAWAL, *Department of Mathematics, Government Science College, Gwalior (M.P.)*

and

H. C. AGRAWAL, *Department of Mathematics, B. K. Post Graduate College, Jhansi (U.P.)*

(*Received* 10 *February* 1975)

In this paper an attempt has been made to find out the operational formulae, recurrence relations, expansions, etc. in case of certain polynomials with the help of difference operators $\triangle$ and E.

§1. In a previous paper Agrawal (1967) has given the operational formulae for the Jacobi, Laguerre and Charlier's polynomials using the operators $\triangle$ and E. The object of this paper is to give the operational formula for the polynomials $A_n^\alpha(x)$ and $M_n^K(x; a, b)$ defined by Srivastava (1964) and Chatterjea (1964) by means of the following relations :

$$\sum_{r=0}^{n} A_r^\alpha(x)\, L_{n-r}^{\alpha+r}(x) = 0; \quad n \geqslant 1 \qquad \ldots(1.1)$$

$$A_0^\alpha(x) = 1 \qquad \ldots(1.2)$$

and

$$M_n^K(x; a, b) = \sum_{r=0}^{n} \binom{n}{r} (Kn + a - K - n + 1)_r \left(\frac{x}{b}\right)^r \qquad \ldots(1.3)$$

In our analysis we shall make the use of a number of known results which we mention here for ready reference

$$\triangle_\alpha f(\alpha) = f(\alpha + 1) - f(\alpha) \qquad \ldots(1.4)$$

$$E_\alpha f(\alpha) = f(\alpha + 1) \qquad \ldots(1.5)$$

$$\triangle_\alpha^n [u_\alpha v_\alpha] = \sum_{r=0}^{n} \binom{n}{r} \triangle_\alpha^{n-r} u_{\alpha+r} \triangle_\alpha^r v_\alpha \qquad \ldots(1.6)$$

and

$$\triangle_{\alpha}^{n} f(\alpha) = \sum_{m=0}^{n} (-)^{n-m} \binom{n}{m} f(\alpha + m). \quad \ldots(1.7)$$

§2. Consider,

$$\frac{(-)^n x^{n+\alpha}}{n! \sqrt{1+\alpha}} \triangle_{\alpha}^{n} \left\{ x^{-\alpha} \sqrt{1+\alpha} \right\} = \frac{(-)^n x^{n+\alpha}}{n! \sqrt{1+\alpha}} \sum_{r=0}^{n} (-)^{n-r} \times \binom{n}{r} x^{-\alpha-r} \sqrt{1+\alpha+r}$$

$$= \frac{1}{n!} \sum_{r=0}^{n} \frac{(-n)_r (1+\alpha)_r}{r!} x^{n-r}.$$

Therefore

$$A_n^{\alpha}(x) = \frac{(-)^n x^{n+\alpha}}{n! \sqrt{1+\alpha}} \triangle_{\alpha}^{n} \left\{ x^{-\alpha} \sqrt{1+\alpha} \right\}. \quad \ldots(2.1)$$

Now operating both the sides of the result [Singh 1964, (1.3)]

$$A_n^{\alpha}(x) = \frac{1}{n!} \sum_{r=0}^{n} \frac{(-n)_r (1+\alpha)_r}{r!} x^{n-r}$$

by the operator $\triangle_{\alpha}$ we get,

$$\triangle_{\alpha} A_n^{\alpha}(x) = (-) A_{n-1}^{\alpha+1}(x). \quad \ldots(2.2)$$

By iteration we have,

$$\triangle_{\alpha}^{r} A_n^{\alpha}(x) = (-)^r A_{n-r}^{\alpha+r}(x). \quad \ldots(2.3)$$

§3. In this section, we shall prove the following result :

$$\sum_{r=0}^{n} \frac{n!}{r!} (-)^r (1+\alpha)_r A_{n-r}^{\alpha+r}(x) \triangle_{\alpha}^{r} = \frac{x^n}{\sqrt{1+\alpha}} [1 - x^{-1} E]^n \sqrt{1+\alpha}$$

$$= (-)^n \prod_{j=1}^{n} [(\alpha - n + 2j)E - x]. \quad \ldots(3.1)$$

PROOF : Let,

$$\Omega_n \phi(\alpha) = \frac{(-)^n x^{n+\alpha}}{\sqrt{1+\alpha}} \triangle_{\alpha}^{n} \left\{ x^{-\alpha} \sqrt{1+\alpha}\, \phi(\alpha) \right\}. \quad \ldots(3.2)$$

Using (1.6), we get

$$\Omega_n \phi(\alpha) = \sum_{r=0}^{n} \frac{n!}{r!} (-)^r (1+\alpha)_r \; A_{n-r}^{\alpha+r}(x). \quad \triangle_\alpha^r \phi(\alpha). \qquad \ldots(3.3)$$

Starting from (3.2) and using the famous result

$$E = 1 + \triangle$$

we derive

$$\Omega_n \phi(\alpha) = \frac{x^n}{\sqrt{1+\alpha}} \left[1 - x^{-1} E \right]^n \sqrt{1+\alpha}\; \phi(\alpha). \qquad \ldots(3.4)$$

Again from (3.2), we have

$$\Omega_n \phi(\alpha) = \frac{(-)^n x^{n+\alpha}}{\sqrt{1+\alpha}} [\alpha \triangle \{\triangle^{n-1} x^{-\alpha} \sqrt{\alpha}\, \phi(\alpha)\}$$

$$+ n E \{\triangle^{n-1} x^{-\alpha} \sqrt{\alpha}\, \phi(\alpha)\}]$$

$$= \frac{(-)^n x^{n+\alpha}}{\sqrt{1+\alpha}} (\alpha \triangle + n E) [\triangle^{n-1} \{x^{-\alpha} \sqrt{\alpha}\, \phi(\alpha)\}]$$

$$= (-)^n \prod_{j=1}^{n} [(\alpha - n + 2j) E - x] \phi(\alpha). \qquad \ldots(3.5)$$

From (3.3), (3.4) and (3.5) we get the required result (3.1).

§4. *Recurrence Relations*—From (2.2), we obtain

$$A_n^{\alpha}(x) = A_n^{\alpha-1}(x) - A_{n-1}^{\alpha}(x). \qquad \ldots(4.1)$$

In (2.1) replacing n by $n+1$ and simplifying it, we establish

$$(n+1) A_{n+1}^{\alpha}(x) = x\, A_n^{\alpha}(x) - (1+\alpha) A_n^{\alpha+1}(x) \qquad \ldots(4.2)$$

above relation is the same as given by Srivastava [1964, (2.9)].

When $\phi(\alpha) = 1$, (3.1) will give

$$n!\, A_n^{\alpha}(x) = (-)^n \prod_{j=1}^{n} [(\alpha - n + 2j) E - x]. \qquad \ldots(4.3)$$

In (4.3) replacing n by $n+1$ and simplifying it, we obtain

$$(n+1) A_{n+1}^{\alpha}(x) = x\, A_n^{\alpha-1}(x) - (\alpha + n + 1) A_n^{\alpha}(x). \qquad \ldots(4.4)$$

Substituting the value of $A_n^{\alpha-1}(x)$ from (4.1) and (4.4), we get,

$$(n+1)\ A_{n+1}^{\alpha}(x) + (n - x + \alpha + 1)\ A_n^{\alpha}(x) = x\ A_{n-1}^{\alpha}(x) \qquad \ldots(4.5)$$

which is (2.8) of Srivastava (1964).

§5. *Summation Results*—In (4.3) replacing n by $n+m$ and using (3.1) and (2.3), we get

$$\binom{m+n}{n} A_{m+n}^{\alpha}(x) = \sum_{r=0}^{\min.(m,n)} \frac{(1+\alpha+n)_r}{r!} A_{m-r}^{\alpha+n+r}(x)\ A_{n-r}^{\alpha-m+r}(x). \qquad \ldots(5.1)$$

Further from (5.1), we have

$$\sum_{n=0}^{\infty} \binom{m+n}{n} t^n A_{m+n}^{\alpha-n}(x) = \sum_{r=0}^{m} \frac{(1+\alpha)_r\, t^r}{r!} A_{m-r}^{\alpha+r}(x) \times \sum_{n=0}^{\infty} t^n A_n^{\alpha-m-n}(x).$$

Now making use of the relations [Singh 1964, (2.5), Srivastava 1964, (5.3)]

$$\sum_{n=0}^{\infty} t^n A_n^{\alpha-n}(x) = (1-t)^{\alpha} \exp[xt/(1-t)]$$

and

$$A_n^{\alpha}(xy) = \sum_{r=0}^{n} \frac{(1+\alpha)_r}{r!} A_{n-r}^{\alpha+r}(x)\, y^{n-r}(1-y)^r$$

we obtain

$$\sum_{n=0}^{\infty} \binom{m+n}{n} t^n A_{m+n}^{\alpha-n}(x) = (1+t)^m (1-t)^{\alpha-m} \exp[xt/(1-t)] \times A_m^{\alpha}[x/(1+t)]. \qquad \ldots(5.2)$$

Repeated application of (4.1) will give,

$$A_{n-K}^{\alpha}(x) = \sum_{r=0}^{K} (-)^{r-K} \binom{K}{r} A_n^{\alpha-r}(x). \qquad \ldots(5.3)$$

Using the famous result

$$f(\alpha+\mu)=\sum_{r}\binom{\mu}{r}\Delta_{\alpha}^{r}f(\alpha),$$

we establish

$$A_{n}^{\alpha+\mu}(x)=\sum_{r=0}^{n}\binom{\mu}{r}(-)^{r}A_{n-r}^{\alpha+r}(x). \qquad \text{...(5.4)}$$

§6. *Expansions*—By putting $\phi(\alpha)=\dfrac{x^{\alpha}}{\sqrt{1+\alpha}}$ in (3.1) and using the result [Erdelyi 1953, eqn. (37)]

$$\Delta_{\alpha}^{r}\left(\frac{x^{\alpha}}{\sqrt{1+\alpha}}\right)=\frac{(-)^{r}\,r!\,x^{\alpha}L_{r}^{\alpha}(x)}{\sqrt{\alpha+r+1}}$$

we get

$$\sum_{r=0}^{n}A_{n-r}^{\alpha+r}(x)\cdot L_{r}^{\alpha}(x)=\frac{x^{n}}{n!}\,{}_1F_0[-n;-;1]. \qquad \text{...(6.1)}$$

Similarly if we put $\phi(\alpha)=\dfrac{1}{\sqrt{1+\alpha}}$ and $\dfrac{x^{-\alpha}}{\sqrt{1+\alpha}}$ we get

$$(1-x)^{n}=(-)^{n}\sum_{r=0}^{n}\frac{n!}{r!}(1+\alpha)_{r}\,A_{n-r}^{\alpha+r}(x)\,{}_1F_1[-r;1+\alpha;1] \qquad \text{...(6.2)}$$

and

$$x^{n}\,{}_1F_0\left[-n;-;\frac{1}{x^{2}}\right]=\sum_{r=0}^{n}\frac{n!}{r!}(1+\alpha)_{r}\,A_{n-r}^{\alpha+r}(x)\,{}_1F_1[-r;1+\alpha;x^{-1}]$$

$$\text{...(6.3)}$$

respectively.

Lastly, by putting $\phi(\alpha)=\dfrac{\sqrt{\alpha+n+1}}{\sqrt{\alpha+1}}$, we observe that

$$\sum_{r=0}^{n}\frac{n!}{r!}(1+\alpha)_{r}\,A_{n-r}^{\alpha+r}(x)\,{}_2F_1[-r,1+\alpha+n;1+\alpha;1]$$

$$=x^{n}\,{}_2F_0[-n,1+\alpha+n,x^{-1}]. \qquad \text{...(6.4)}$$

§7. In a similar way the following results can be obtained in case of the polynomials $M_n^K(x; a, b)$.

$$M_n^K(x; a, b) = \frac{(-)^{-n-a}\left(\frac{x}{b}\right)^{-a}}{\sqrt{Kn+a-K-n+1}} \triangle_a^n \left[\left(-\frac{x}{b}\right)^a \sqrt{Kn+a-K-n+1} \right] \quad ...(7.1)$$

$$\triangle_a M_n^K(x; a, b) = \left(\frac{nK}{b}\right) M_{n-1}^K(x; a+K, b) \quad ...(7.2)$$

$$\triangle_a^r M_n^K(x; a, b) = (-n)_r \left(-\frac{x}{b}\right)^r M_{n-r}^K(x; a+rK, b) \quad ...(7.3)$$

$$\sum_{r=0}^{n} \binom{n}{r} (Kn+a-K-n+1)_r \left(\frac{x}{b}\right)^r M_{n-r}^K(x; a+rK, b) \triangle_a^r$$

$$= \frac{1}{\sqrt{Kn+a-K-n+1}} \left[\frac{x}{b} E + 1\right]^n \sqrt{Kn+a-K-n+1}$$

$$= \prod_{j=1}^{n} \left[\left(\frac{x}{b}\right)(Kn+a-K-2n+2j) E + 1\right] \quad ...(7.4)$$

$$\prod_{j=1}^{n} \left[\left(\frac{x}{b}\right)(Kn+a-K-2n+2j) E + 1\right] = M_n^K(x; a, b) \quad ...(7.5)$$

$$M_{m+n}^K(x; a, b) = \sum_{r=0}^{\min(m,n)} \binom{m}{r} (Km+a+Kn-K-m+1)_r \left(\frac{x}{b}\right)^{2r}$$

$$\times\, r! \binom{n}{r} M_{m-r}^K(x; a+Kn+rK, b)\, M_{n-r}^K(x; a+Km+rK-2m, b) \quad ...(7.6)$$

$$M_{n+1}^K(x; a, b) = \frac{x}{b}(Kn+a-n) M_n^K(x; a+K, b) + M_n^K(x; a+K-1, b) \quad ...(7.7)$$

$$M_n^K(x; a, b) = M_n^K(x; a-1, b) + n\left(\frac{x}{b}\right) M_{n-1}^K(x; a+K-1, b) \quad ...(7.8)$$

$$M_{n+1}^K(x; a, b) = \frac{x}{b}(a+Kn) M_n^K(x; a+K-1; b)$$

$$+ M_n^K(x; a+K-2, b) \quad ...(7.9)$$

$$b^2 [M_{n+1}^K (x; a, b) - M_n^K (x; a + K - 2, b)] = x(Kn + a) [b M_n^K \times (x, a + K - 2, b) + nx M_{n-1}^K (x; a + 2K - 2, b)] \quad ...(7.10)$$

$$\left(1 + \frac{x}{b}\right)^n = \sum_{r=0}^{n} \binom{n}{r} (Kn + a - K - n + 1)_r \left(-\frac{x}{b}\right)^r M_{n-r}^K \times (x; a + rK, b) \times {}_1F_1 \left[\begin{matrix} -r; \\ Kn + a - K - n + 1, \end{matrix} 1\right] \quad ...(7.11)$$

$$\sum_{r=0}^{n} \binom{n}{r} (Kn + a - K - n + 1)_r \left(-\frac{x}{b}\right)^r M_{n-r}^K (x; a + rK, b) \times {}_2F_1 \left[\begin{matrix} -r, a + 1; \\ Kn + a - K - n + 1; \end{matrix} \frac{x}{b}\right] = {}_2F_0 \left[\begin{matrix} -n, a + 1; \\ \text{———} ; \end{matrix} -\frac{x^2}{b^2}\right] \quad ...(7.12)$$

and

$$\sum_{r=0}^{n} \binom{n}{r} r! \frac{(Kn + a - K - n + 1)_r}{(a + 1)_r} \left(-\frac{x}{b}\right)^r M_{n-r}^K (x; a + rK, b) L_r^a (x) = {}_2F_1 \left[\begin{matrix} -n, Kn + a - K - n + 1; \\ a + 1; \end{matrix} -\frac{x^2}{b}\right]. \quad ...(7.13)$$

Particularly for $K = 2$, (7.9), (7.10) and (7.6) reduces to the well known results [Agrawal 1954, eqn. (8); Chatterjea 1963, eqns. (3.8) and (3.17) respectively].

References

Agrawal, B. M. (1967). Operational formulae for classical polynomials. *Vijnana Parishad Anusandhan Patrika*, **10**, 43-50.

Agrawal, R. P. (1954). On Bessel polynomials. *Can. J. math.*, **6**, 410-15.

Chatterjea, S. K. (1964). A generalization of the Bessel polynomials. *Mathematica*, **6**, 19-29.

——— (1963). Operational formula for certain classical polynomials. *Quart. J. Math., Oxford Ser.* (2) **14**, 241-46.

Erdelyi, A. (1953). Higher Transcendental Functions, Vol. II. McGraw-Hill Book Co. Inc., New York.

Singh, R. P. (1964). Some polynomial of Srivastava related to the Laguerre polynomial. *Math. Japon.*, **9**, 5-9.

Srivastava, K. N. (1964). Some polynomials related to the Laguerre polynomials. *J. Indian math. Soc.*, **28** (2), 43-50.