EFFECT OF VISCOSITY ON THE SURFACE WAVES PRODUCED BY AN EXPLOSION ABOVE A LIQUID

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The three-demensional problem of surface waves caused by the interaction of a blast wave with a heavy (or light) viscous liquid has been investigated. The resulting integral for the surface displacement has been analysed for general as well as special pressure distributions on the hypothesis of a small viscosity. Asymptotic expression for the surface displacement has been derived for large values of the time at large distances from the advancing pressure area. Some interesting points of the motion have been discussed.

INTRODUCTION

The problem of surface waves generated by the interaction of a blast wave due to an explosive source situated above a liquid, with the liquid has been well-investigated by Rumiantsev (1960), Kisler (1960) and Sen (1963) on the hypothesis that the liquid is frictionless.

In this problem, it is assumed, because of the wide difference between the densities of the gas and the liquid, that the liquid displacements are too small to influence the motion of the gas, which is assumed to be known. The problem thus becomes one of infinitesimal wave motion due to a known unsteady pressure distribution over a time-varying area on the free surface of the liquid.

It is our object in this paper to formulate the problem stated above for the case of a deep heavy liquid in three dimensions, taking into account the viscosity of the liquid.

The analysis of the resulting integral for surface displacement, admittedly a tough one, is carried out for a general pressure distribution on the hypothesis of a small viscosity and its series representations together with expressions suitable for a numerical quadrature are derived. The integration in the series expressions has been completed in full for some special forms of pressure distributions.

Assuming a pressure field closely resembling the one inside a blast wave, we have derived an asymptotic expression for the surface displacement for large values of the time at large distances from the advancing pressure area. Lastly, we deduce

similar expressions, for a weightless fluid and end with a discussion of some points of interest.

1. Formulation of the Problem

We consider an infinite ocean of viscous incompressible fluid occupying the space $-\infty<z\leqslant 0$, $-\infty<x<\infty$, $-\infty<y<\infty$, which is at rest at time $t=0$.

The axes of x and y are taken on the undisturbed horizontal free surface and the z-axis is vertically upward.

Surface waves are generated when a spherical shock wave due to a point blast in air interacts with the water surface, thus giving rise to an expanding circular region of pressure on the surface. We pose the problem of finding the form of the free fluid surface at any time $t>0$.

In cylindrical coordinates (r, θ, z), the pressure function $p_0(r, t)$ is given by

$$\left.\begin{aligned} H(t)\,p_0(r,t)&=F(r,t) && \text{when} \quad r<r_0(t)\\ &=0 && \text{when} \quad r>r_0(t)\end{aligned}\right\},\quad t>0,\ z=0; \qquad \ldots(1.1)$$

where

$$\begin{aligned} H(t)&=0 && \text{when} \quad t<0\\ &=1 && \text{when} \quad t>0.\end{aligned}$$

For slow motions symmetrical about $0z$, Stokes-Navier equations of motion and the equation of continuity are

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p_1}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 w}{\partial r\partial z}\right) \qquad \ldots(1.2)$$

$$\frac{\partial w}{\partial t}=-g-\frac{1}{\rho}\frac{\partial p_1}{\partial z}+\nu\left(-\frac{\partial^2 u}{\partial r\partial z}-\frac{1}{r}\frac{\partial u}{\partial z}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right), \qquad \ldots(1.3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru\right)+\frac{\partial w}{\partial z}=0. \qquad \ldots(1.4)$$

The initial and the boundary conditions are

$$u=w=0,\ \zeta=0 \text{ at } t=0 \qquad \ldots(1.5)$$

$$\frac{\partial \zeta}{\partial t}=\left(w\right)_{z=0} \qquad \ldots(1.6)$$

Let $p=p_1+g\rho z$. ... ...(1.7)

We have on $z=0$,

$$-p_0=p_{nn}\simeq p_{zz}=-p_1+2\mu\frac{\partial w}{\partial z}=-p+g\rho\zeta+2\mu\frac{\partial w}{\partial z},$$

$$0=p_{n\tau_1}\simeq p_{zr}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),$$

$$0=p_{n\tau_2}\simeq p_{z\theta}=0. \qquad \ldots(1.8)$$

At infinity, $u=w=0.$ $\ldots(1.9)$

As $r\to\infty$, u, w, p and their r-derivatives are zero.

Here n, τ_1, τ_2 refer to the normal and two orthogonal directions on the surface; (u, o, w) are the components of velocity in the increasing directions of (r, θ, z).

2. Solution

We seek to obtain the exact integral expression for the surface displacement ζ.

Let $(\bar{u}, \bar{w}, \bar{p}, \bar{\zeta}, \bar{p}_0)$ denote the Hankel transforms of (u, w, p, ζ, p_0) in r and (U, W, P, E, P_0) denote the Laplace transforms of the former in t; k and s being the respective transform parameters. For instance,

$$\bar{u}(k, z, t)=\int_0^\infty u(\alpha, z, t)\,\alpha\, J_0(k\alpha)\,d\alpha,$$

$$U(k, z, s)=\int_0^\infty \bar{u}(k, z, t)\,e^{-st}dt.$$

By (1.2) to (1.4)

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial z^2}=0.$$

Transforming this equation suitably, we get

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)P(k, z, s)=0. \qquad \ldots(2.1)$$

The solution of (2.1) is

$$P=A(k, s)\,e^{kz}+B(k, s)e^{-kz}$$

Since the pressure p is bounded at $z=-\infty$, we put $B(k, s)=0$.

Therefore $P=A(k, s)\,e^{kz}$ $\ldots(2.2)$

Eliminating u from (1.2) and (1.3), we have by virtue of (1.4)

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right). \qquad \ldots(2.3)$$

Transforming (2.3) and eqns. (1.5)—(1.9) suitably, we get,

$$\frac{\partial^2 W}{\partial z^2}-\left(\frac{s}{\nu}+k^2\right)W=\frac{kP}{\nu\rho}=\frac{Ak}{\nu\rho}e^{kz} \qquad (\ldots 2.4)$$

$$\left.\begin{aligned}&\bar{u}=\bar{w}=0,\ \bar{\zeta}=0 \text{ for } t=0\\ &U=W=0 \text{ at } z=-\infty\\ &sE=W(k, 0, s)\\ &P_3=P-g\rho E-2\mu\frac{\partial W}{\partial z} \text{ on } z=0.\end{aligned}\right\} \qquad (\ldots 2.5)$$

Differentiating (1.4) with respect to z and applying the second relation of (1.8), we get

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{\partial^2 w}{\partial z^2} = 0 \text{ on } z=0.$$

This is transformed into

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) W = 0 \quad \text{on } z=0. \qquad \ldots(2.6)$$

The solution of (2.4) which vanishes at $z=-\infty$ is

$$W = C(k,s)\, e^{az} - \frac{Ak}{s\rho} e^{kz}$$

where $$a^2 = \frac{s}{\nu} + k^2 \qquad \ldots(2.7)$$

By (2.6) $$(k^2+a^2)C - \frac{2Ak^3}{s\rho} = 0. \qquad \ldots(2.8)$$

By (2.7) and the third and the fourth relations of (2.5), we get

$$P_0 = A\left[1 + \frac{gk}{s} + \frac{2\mu k^2}{s\rho}\right] - C\left[\frac{g\rho}{s} + 2\mu a\right]. \qquad \ldots(2.9)$$

Solving (2.8) and (2.9), we get

$$C = \left(\frac{2\nu k^3}{\rho}\right)\frac{P_0}{\Delta}, \qquad A = s(s+2\nu k^2)\, P_0/\Delta \qquad \ldots(2.10)$$

where

$$\Delta \equiv \Delta(k, s) = \left(s+2\nu k^2\right)^2 - 4\nu^{3/2} k^3 \left(s+\nu k^2\right)^{\frac{1}{2}} + gk.$$

The values of A and C, when substituted in (2.7), give for W the value

$$W = kP_0 (\rho\Delta)^{-1}\left[2\nu k^2 \exp\left(z\sqrt{s/\nu + k^2}\right) - (s+2\nu k^2)\exp(kz)\right] \qquad \ldots(2.11)$$

The third relation of (2.5) now gives

$$E = \lim_{z\to 0-} kP_0 (\rho\Delta s)^{-1}\left[2\nu k^2 \exp\left(z\sqrt{s/\nu + k^2}\right) - (s+2\nu k^2)\exp(kz)\right]$$

By the inversion theorem for Laplace transforms, we then have

$$\bar{\zeta} = \frac{k}{2\pi i\rho} \lim_{z\to 0} \int_0^t \bar{p}_0(k,\tau)\, d\tau \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-\tau)}}{s\Delta(k,s)}\left[2\nu k^2 \exp\left(z\sqrt{s/\nu + k^2}\right) - (s+2\nu k^2)\exp(kz)\right] ds \qquad \ldots(2.12)$$

the last result following from the convolution theorem. In the above, it suffices to take $\gamma \geqslant 0$ as the poles of the s-integral have negative real parts.

This gives, by Hankel's inversion theorem,

$$\zeta = \int_0^\infty \bar{\zeta}\, k J_0\,(kr)\, dk. \qquad \ldots(2.13)$$

Observing that

$$\bar{p}_0\,(k,\ \tau) = \int_0^{r_0(\tau)} F\,(\alpha,\ \tau)\ \alpha J_0\,(k\alpha)\ d\alpha,$$

we obtain finally,

$$\zeta = \frac{1}{2\pi i \rho} \lim_{z \to 0-} \int_0^\infty k^2 J_0\,(kr)\ dk \int_0^t d\tau \int_0^{r_0(\tau)} F\,(\alpha,\ \tau)\ \alpha J_0\,(k\alpha)\ d\alpha \times$$

$$\times \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-\tau)}}{s\Delta(k,\ s)} \left[2\nu k^2 \exp\left(z \sqrt{s/\nu + k^2} \right) - \left(s + 2\nu k^2 \right) \exp\,(kz) \right] ds. \qquad \ldots(2.14)$$

3. Approximate Expression for ζ for small Values of ν

It will now be proved that when $\bar{p}_0\,(k,\ t)$ is $O\,(t^{-1})$, $t \to \infty$, the last expression (2.14) may be replaced by the following:

$$\zeta = -\frac{1}{2\pi i \rho} \int_0^\infty k^2 J_0\,(kr)\, dk \int_0^t d\tau \int_0^{r_0(\tau)} \alpha F\,(\alpha,\ \tau)\ J_0\,(k\alpha)\, d\alpha$$

$$\times \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-\tau)}}{(s + 2\nu k^2)^2 + gk}\, ds + O\left(\nu^{3/2}\right). \qquad \ldots(3.1)$$

In the first place, we shall examine the uniform convergence of the part of the s-integral with $\frac{1}{\Delta}$ in the integrand of (2.14) replaced by $\Big\{(4k^3\nu^{3/2})^{-1} \times$ the second term of the indentity below $\Big\}$ in $0 \leqslant \nu \leqslant \nu_0 < 1$.

$$\frac{1}{\Delta(k,\ s)} = \frac{1}{(s + 2\nu k^2)^2 + gk} + \frac{4k^3\nu^{3/2}(s + \nu k^2)^{1/2}}{[(s + 2\nu k^2)^2 + gk]\Delta} \qquad \ldots(3.2)$$

If $s=\gamma+iy$, $r_1\cos\theta_1=\gamma+\nu k^2$, $r_1\sin\theta_1=y$, one gets as $\nu\to 0_+$

$$\frac{1}{2}\frac{\partial}{\partial\nu}\left|\triangle(k,\ s)\right|^2=\frac{1}{2}\frac{\partial}{\partial\nu}\left[\left\{(\gamma+2\nu k^2)^2-y^2+gk-4k^3\nu^{3/2}r_1^{1/2}\cos\frac{\theta_1}{2}\right\}^2\right.$$
$$\left.+\left\{2y\,(\gamma+2\nu k^2)-4k^3\nu^{3/2}r_1^{1/2}\sin\frac{\theta_1}{2}\right\}^2\right]$$
$$=[(\gamma^2-y^2+gk).4k^2\gamma+(2y\gamma.4k^2y)]$$
$$=(\gamma^2+y^2+gk).4k^2\gamma>0$$
for $\gamma>0$ except when $k=0$.

Therefore, $|\triangle(k,s)|$ increases with ν in a right neighbourhood of $\nu=0$. Thus

$$|\triangle(k,s)|\geqslant|\triangle(k,s)|_{\nu=0}=|s^2+gk| \qquad \text{when} \qquad 0\leqslant\nu\leqslant\nu_0<1.$$

Further, since $z\leqslant 0$,

$$|2\nu k^2\exp\left(\frac{s}{\nu}+k^2\right)^{1/2}z-(s+2\nu k^2)\exp(kz)|\leqslant 2k^2+|s+2k^2|.$$

Therefore

$$\left|\frac{(s+\nu k^2)^{1/2}e^{s(t-\tau)}\left[2\nu k^2\exp\left(\frac{s}{\nu}+k^2\right)^{1/2}z-(s+2\nu k^2)\exp(kz)\right]}{s\{(s+2\nu k^2)^2+gk\}\triangle(k,s)}\right|$$
$$\leqslant\frac{\{2k^2+|s+2k^2|\}|s+k^2|^{1/2}e^{\gamma(t-\tau)}}{|s|\ |s^2+gk|^2},\quad -\infty<\operatorname{Im}s<\infty.$$

The uniform convergence of the part of the s-integral as stated above in $0\leqslant\nu\leqslant\nu_0<1$ now follows by the analogue of the M-test. Hence we may write

$$\lim_{\nu\to 0+}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\left[2\nu k^2\exp\left(\frac{s}{\nu}+k^2\right)^{1/2}z-(s+2\nu k^2)\exp(kz)\right](s+\nu k^2)^{1/2}\,e^{s(t-\tau)}ds}{s\{(s+2\nu k^2)^2+gk\}\triangle(k,\ s)}$$
$$=-\frac{e^{kz}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^{1/2}e^{s(t-\tau)}}{(s^2+gk)^2}ds$$
$$=-e^{kz}(t-\tau)^{5/2}{}_1F_2\left(2;\frac{7}{4},\frac{9}{4};-\frac{1}{4}gk\,(t-\tau)^2\right)\Big/\Gamma\left(\frac{7}{2}\right) \qquad \text{.. (3.3)}$$

(Erdélyi 1954, p. 238 (10), Vol. I).

For any fixed value of τ in $0\leqslant\tau\leqslant t-t\delta$, $(0<\delta<1)$, we know that (Luke 1962, pp. 4-11) as $\frac{1}{4}gk\,(t-\tau)^2\to\infty$, the last expression is asymptotically equal to

$$\frac{e^{kz}(t-\tau)}{2\sqrt{2}}(gk)^{-3/4}\left\{\cos\sqrt{gk}\,(t-\tau)-\sin\sqrt{gk}\,(t-\tau)\right\}. \qquad \text{...(3.4)}$$

Let $\bar{\zeta}_0$ and ζ_0 denote respectively the part of $\bar{\zeta}$ and ζ obtained by replacing $\triangle^{-1}$ in (2.12) and (2.14) by the second term on the right of (3.2). Then, by the first mean value theorem of integral calculus,

$$\lim_{\nu\to 0_+} \nu^{-3/2}\bar{\zeta}_0 \simeq \frac{2\sqrt{2}}{7}\rho^{-1}g^{-3/4}k^{13/4}t^2(1-\delta)^{-3/2}\bar{p}_0(k,\delta t) \times \{\cos\sqrt{gk}\,t(1-\delta)-\sin\sqrt{gk}\,t(1-\delta)\}\, e^{kz}.$$

In ζ_0, we use this result and also replace $J_0(kr)$ by the dominant term of its asymptotic expansion. This gives,

$$\lim_{\nu\to 0+}\nu^{-3/2}\zeta_0 = \frac{4\rho^{-1}}{7\sqrt{\pi r}}g^{-3/4}(1-\delta)^{-3/2}t^2 \lim_{z\to 0-}\int_0^\infty k^{15/4}\bar{p}_0(k,\delta t)\cos\left(kr-\frac{\pi}{4}\right) \times \{\cos\sqrt{gk}\,t(1-\delta)-\sin\sqrt{gk}\,t(1-\delta)\}e^{kz}dk.$$

We observe that the last integral is absolutely and uniformly convergent in $z\leqslant 0$ if $\bar{p}_0(k,t)=O(k^{-19/4-m'})$, $m'>0$ as $k\to\infty$ so that the limiting process $z\to 0-$ may be dispensed with in that case. By applying the method of stationary phase for large t, we also find that this integral is bounded as $t\to\infty$ if we assume $t\dot{\bar{p}}_0(k,t)$ is bounded as $t\to\infty$.

Thus for small values of ν,

$$\zeta_0 = O(\nu^{3/2}).$$

In the remaining part $\zeta-\zeta_0$ of (2.14), the α-integral and the s-integral (section 4) are at least $O(k^{-3/2})$ and $O(k^{-1/2})$, as $k\to\infty$, respectively so that the multiple integral converges uniformly in $z\leqslant 0$. We can therefore directly put $z=0$ in $\zeta-\zeta_0$ of (2.14) and hence (3.1) follows. Henceforth we shall regard ζ_0 as negligibly small and designate $\zeta-\zeta_0$ as ζ.

4. Evaluation of the s- and k-Integrals of ζ

(A) *Expressions of ζ in a Closed Form*

The adoption of the hypothesis of small viscosity helps us to complete the s- and k-integrations of ζ in (3.1), and to express the result in a closed form involving known functions. The resulting expression for ζ may be subjected to a numerical integration.

First, we consider the s-integral in ζ of (3.1):

$$\zeta^{(s)} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{s(t-\tau)}ds}{(s+2\nu k^2)^2+gk}.$$

The s-integrand is regular in the s-plane except for the two poles

$$s_j = -2\nu k^2 + (-1)^j i\sqrt{gk}, \; j=1, 2.$$

Integrating along a usual contour enclosing the poles and involving Cauchy's residue theorem, we get

$$\zeta^{(s)} = \sum_{j=1}^{2} \frac{(-1)^j e^{s_j(k)(t-\tau)}}{2i\sqrt{gk}}$$

$$= \frac{e^{-2k^2\nu(t-\tau)}}{\sqrt{gk}} \sin \sqrt{gk}\,(t-\tau).$$

Now

$$\bar{p}_0(k, t) = r_0^2(t) \int_0^1 F(\alpha r_0(t), t)\; \alpha J_0(k\alpha r_0(t))\; d\alpha$$

Therefore,

$$\zeta = -\frac{1}{\rho\sqrt{g}} \int_0^1 \alpha d\alpha \int_0^t r_0^2(\tau)\; F(\alpha r_0(\tau), \tau) d\tau \int_0^\infty k^{3/2} J_0(kr) J_0(kr_0(\tau)\alpha)$$
$$\times\; e^{-2\nu k^2(t-\tau)} \sin \sqrt{gk}\,(t-\tau) dk.$$

If we suppose that the functions F and r_0 are such as to ensure the convergence of ζ and $\frac{\partial \zeta}{\partial \nu}$ for $\nu=0$, we may introduce (McLachlan 1954) a factor e^{-ck} in the integrand above and obtain the value of ζ by a passage to the limit, $c \to 0_+$. of the whole expression so formed.

A deduction from Neumann's addition theorem for Bessel functions gives

$$\pi J_0(kr) J_0(k\alpha r_0(\tau)) = \int_0^\pi J_0(k\lambda) d\theta \qquad \ldots(4.1)$$

where $\lambda^2 = r^2 + \alpha^2 r_0^2(\tau) - 2\alpha r r_0(\tau) \cos\theta.$ $\qquad \ldots(4.2)$

Therefore,

$$\pi g \rho \zeta = \lim_{c \to 0_+} \int_0^1 \alpha d\alpha \int_0^t r_0^2(\tau)\; F(\alpha r_0(\tau), \tau) d\tau \int_0^\pi d\theta \int_0^\infty \sqrt{g}\, k^{3/2} J_0(k\lambda)$$
$$\times\; e^{-ck - 2\nu k^2(t-\tau)} \sin \sqrt{gk}\,(t-\tau) dk. \qquad \ldots(4.3)$$

In the above, we have performed a permissible change of order of k- and θ-integrals.

Correct to the first power of ν, (4.3) is

$$-\pi g\rho\zeta = \int_0^1 \alpha d\alpha \int_0^t r_0^2(\tau)\, F(\alpha r_0(\tau),\tau)d\tau \int_0^\pi d\theta\, [\zeta_0^{(k)} - (t-\tau)\zeta_1^{(k)}] \quad \ldots(4.4)$$

where

$$\zeta_{0,1}^{(k)} = \lim_{c\to 0_+} \int_0^\infty g^{1/2}\{k^{3/2}, k^{7/2}\}\, J_0(k\lambda)\, \sin\sqrt{gk}\,(t-\tau)\, e^{-ck}dk. \quad \ldots(4.5)$$

The function $\zeta_0^{(k)}$ defined in this way corresponds to the k-integral in the expression for the surface displacement in a non-viscous fluid, and is thus obtained from Sen's paper (1963). With a slight change in notations, we thus have

$$\zeta_0^{(k)} = -\left(\frac{\pi}{8\sqrt{2}}\right) g(t-\tau)\lambda^{-3}\,[(1-8x_1^2)\, J_{1/4}J_{-1/4} + 6x_1(J_{-3/4}J_{-1/4} - J_{3/4}J_{1/4})$$

$$-8x_1^2 J_{3/4}J_{-3/4}] \quad . \quad (4.6)$$

where

$$x_1 = g(t-\tau)^2/8\lambda \quad \ldots(4.7)$$

is the common argument of all the Bessel functions written here.

The function $\zeta_1^{(k)}$ may also be evaluated by following the process used for calculating $\zeta_0^{(k)}$. First, we replace $\sin\sqrt{gk}\,(t-\tau)$ in $\zeta_1^{(k)}$ by its power series expansion, and interchange the operations of summation and k-integration to obtain:

$$\zeta_1^{(k)} = \lim_{c\to 0_+} \sum_{n=0}^\infty \frac{(-1)^n g^{n+1}(t-\tau)^{2n+1}}{(2n+1)!} \int_0^\infty k^{n+4}e^{-ck}J_0(k\lambda)dk.$$

By using the integral representation [Erdélyi 1953, Vol. II, p. 49 (16)]

we have

$$\int_0^\infty k^{n+4}e^{-ck}J_0(k\lambda)\, dk = \Gamma(n+5)\, (c^2+\lambda^2)^{-1/2(n+5)}\, {}_2F_1\left(\frac{n+5}{2}, -\frac{n+4}{2};\, 1;\, \frac{\lambda^2}{\lambda^2+c^2}\right).$$

When $c=0$, Gauss value for ${}_2F_1\,(a;b;1;1)$ gives

$${}_2F_1\left(\frac{n+5}{2}, -\frac{n+4}{2};\, 1;\, 1\right) = \frac{\sqrt{\pi}}{\Gamma\left(1-\frac{n+5}{2}\right)\Gamma\left(\frac{n+6}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{n+5}{2}\right)\sin\left(\frac{n+5}{2}\right)\pi}{\Gamma\left(\frac{n+6}{2}\right)\sqrt{\pi}}.$$

This gives

$$\lim_{c\to 0_+} \int_0^\infty k^{n+4} e^{-ck} J_0(k\lambda)\, dk$$

$$= \pi^{-1/2} \frac{\Gamma(2m+5)\Gamma(m+\frac{5}{2})}{\Gamma(m+3)} (-1)^m \lambda^{-(2m+5)}$$

when $n=2m$, $m=0, 1, 2, 3, \ldots$ ....,

$=0$ when $n=2m+1$.

Therefore

$$\zeta_1^{(k)} = \frac{\pi^{-1/2} g(t-\tau)}{\lambda^5} \sum_{n=0}^{\infty} (-1)^n \left\{\frac{g(t-\tau)^2}{\lambda}\right\}^{2n} \times$$

$$\times \frac{(2n+4)!\, \Gamma(n+\frac{5}{2})}{(4n+1)!\ (n+2)!} \qquad \ldots(4.8)$$

Using duplication formula for gamma functions, we get from above

$$\zeta_1^{(k)} = \frac{\pi g(t-\tau)}{\sqrt{2}\,\lambda^5} \sum_{n=0}^{\infty} (-1)^n \left\{\frac{g(t-\tau)^2}{16\lambda}\right\}^{2n} \frac{(2n+3)^2\ (2n+1)^2 \Gamma(2n+1)}{n!\ \Gamma(n+1)\ \Gamma(n+\frac{3}{4})\ \Gamma(n+\frac{5}{4})}.$$

Applying Schläfli and Schönholzer's series formula (Whittaker and Watson 1952, p. 380) for products of Bessel functions, we express the last series in a closed form. This gives finally

$$\zeta_1^{(k)} = \left(\frac{\pi}{8\sqrt{2}}\right) g(t-\tau)\lambda^{-5}\left[\left(\frac{25}{4} - 254x_1^2 + 32x_1^4\right) J_{1/4} J_{-1/4}\right.$$

$$\left. + \left(\frac{119}{2} x_1 - 168x_1^3\right)(J_{-3/4} J_{-1/4} - J_{3/4} J_{1/4}) + (32x_1^4 - 178x_1^2)\, J_{3/4} J_{-3/4}\right]$$

$\ldots(4.9)$

where x_1 is again the same common argument of all Bessel functions written here.

Returning to (4.4), we now rewrite the expression for ζ in a form in which a numerical integration of the θ-integral is possible as a two-parameter function which is not affected by the actual law of pressure changes. We define these parameters A_1 and B_1 as follows :

$$A_1 = g(t-\tau)^2/8[r+\alpha r_0(\tau)], \qquad B_1 = 4rr_0(\tau)\alpha(r+\alpha r_0(\tau))^{-2} \qquad \ldots(4.10)$$

Writing $C_1 = (1-B_1\cos^2\theta)^{1/2}$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$, we find that in the transformation $\theta \to 2\theta$,

$$x_1=\frac{A_1}{C_1}, \qquad \lambda=C_1[r+\alpha r_0(\tau)],$$

$$\zeta_0^{(k)}=-\frac{\pi}{8\sqrt{2}}\, g(t-\tau)(r+\alpha r_0(\tau))^{-3}C_1^{-3}[\{1-8(A_1/C_1)^2\}\, J_{1/4}J_{-1/4}+$$

$$+6(A_1/C_1)(J_{-3/4}J_{-1/4}-J_{3/4}J_{1/4})-8(A_1/C_1)^2J_{3/4}J_{-3/4}]$$

$$\zeta_1^{(k)}=\frac{\pi}{8\sqrt{2}}\, g(t-\tau)[r+\alpha r_0(\tau)]^{-5}C_1^{-5}\left[\left\{\frac{25}{4}-254(A_1/C_1)^2+\right.\right.$$

$$\left.+32(A_1/C_1)^4\right\}J_{1/4}J_{-1/4}+\frac{A_1}{C_1}\left\{\frac{119}{2}-168(A_1/C_1)^2\right\}(J_{-3/4}J_{1/4}-J_{3/4}-J_{1/4})$$

$$\left.+\left(\frac{A_1}{C_1}\right)^2\left\{32\left(\frac{A_1}{C_1}\right)^2-178\right\}J_{3/4}J_{-3/4}\right] \qquad \ldots(4.11)$$

where A_1/C_1 is the common argument of all the Bessel functions written above. The range of θ-integration being now halved, ζ from (4.4) goes into the form

$$-\pi g\rho\zeta=2\int_0^1 \alpha d\alpha \int_0^t r_0^2(\tau)\; F\;(\alpha r_0(\tau),\tau)\; d\tau \int_0^{\pi/2}[\zeta_0^{(k)}-2\nu(t-\tau)\zeta_1^{(k)}]\, d\theta,\ \theta\to 2\theta. \quad \ldots(4.12)$$

The actual numerical integration of the θ-integral will not be attempted here.

(B) Development of ζ in an Infinite Series

It is possible to complete up to the θ-integration in (4.4) in an infinite series of hypergeometric functions and thus obtain a mixed integro-series expression for ζ. We shall confine our attention only to the part of ζ involving $\zeta_1^{(k)}$ as discussion of the other part may be obtained from Sen's (1963) paper.

For this purpose, we integrate the series (4.8) term by term with respect to θ over $(0, \pi)$, and use the result

$$\int_0^\pi \left[1+\frac{\alpha^2r_0^2(\tau)}{r^2}-\frac{2\alpha r_0(\tau)}{r}\cos\theta\right]^{-n-\frac{5}{2}}d\theta$$

$$=\pi\, {}_2F_1\left(n+\tfrac{5}{2}, n+\tfrac{5}{2};\ 1;\ \frac{\alpha^2r_0^2(\tau)}{r^2}\right),\ r>r_0(t).$$

Simplifying also the factorials in (4.8) by using the duplication formula for gamma functions, we get

$$\int_0^\pi \zeta_1^{(k)}d\theta=16g\,(t-\tau)r^{-5}\sum_0^\infty (-1)^n\left\{\frac{2g(t-\tau)^2}{r}\right\}^{2n}\frac{\{\Gamma(n+\frac{5}{2})\}^2}{(4n+1)!}\times$$

$$\times\ {}_2F_1\left(n+\tfrac{5}{2}, n+\tfrac{5}{2};\ 1;\ \frac{\alpha^2r_0^2(\tau)}{r^2}\right),\ r>r_0(t) \qquad \ldots(4.13)$$

The last series is absolutely convergent for all values of $gr^{-1}(t-\tau)^2$, $r>r_0(t)$. For using the relations between contiguous hypergeometric series [Erdélyi 1953, 2.8 (35, 38, 41)], we have the result

$$\frac{{}_2F_1(a'+1,\ a'+1;\ 1;\ z)}{{}_2F_1(a',\ a';\ 1;\ z)} = \frac{a'+z(3a'-2)}{a'(1-z)^2} + \frac{2z(a'-1)^2\ {}_2F_1(a',\ a';\ 2;\ z)}{a'(1-z)^2\ {}_2F_1(a',\ a';\ 1;\ z)}.$$

Our proposition now follows by an application of D' Alembert's ratio test.

Using (4.13) in (4.4) we get,

$$\zeta=\zeta_{\nu=0}+32\nu(\pi\rho)^{-1}r^{-5}\int_0^1 \alpha d\alpha \int_0^t r_0^2(\tau)F(\alpha r_0(\tau),\tau)(t-\tau)^2$$

$$\times\sum_{n=0}^{\infty}(-1)^n\left\{\frac{2g(t-\tau)^2}{r}\right\}^{2n}\frac{\{\Gamma(n+\frac{5}{2})\}^2}{(4n+1)!}\ {}_2F_1\left(n+\tfrac{5}{2},\ n+\tfrac{5}{2};\ 1;\ \frac{\alpha^2r_0^2(\tau)}{r^2}\right)d\tau,$$

$$r>r_0(t). \qquad \ldots(4.14)$$

The term $\zeta_{\nu=0}$ is the value of ζ in a non-viscous fluid and is found from Sen's paper. We are now left with, namely, those with respect to α and τ. To complete these, we need know the actual law of pressure variation as also the function $r_0(t)$.

(B_1) *Illustrative Cases*

(i) Let

$$F(r,\ t)=\lambda_m(t)r^m,\ (m>-\tfrac{3}{2}) \qquad \ldots(4.15)$$

The restriction mentioned is sufficient for the convergence of the α-integral near $\alpha=0$, an additional factor $\alpha^{-1/2}$ occurring when $J_0(k\alpha)$ is replaced by $O((k\alpha)^{-1/2})$ near $k=\infty$.

It is known that (Erdélyi 1954, § 6.2)

$$\int_0^1 \alpha^{m+1}\ {}_2F_1\left(n+\frac{5}{2},\ n+\frac{5}{2};\ 1,\ \frac{\alpha^2r_0^2(\tau)}{r^2}\right)d\alpha$$

$$=(m+2)^{-1}\ {}_3F_2\left(n+\frac{5}{2},\ n+\frac{5}{2},\ \frac{m}{2}+1;\ 1,\ \frac{m}{2}+2;\ \frac{r_0^2(\tau)}{r^2}\right)$$

where ${}_3F_2$ denotes a generalized hypergeometric series. Therefore,

$$\zeta=\zeta_{\nu=0}+32\nu\rho^{-1}(m+2)^{-1}\pi^{-1}r^{-5}\int_0^t \lambda_m(\tau)[r_0(\tau)]^{m+2}(t-\tau)^2d\tau$$

$$\times\sum_{n=0}^{\infty}(-1)^n\left\{\frac{2g(t-\tau)^2}{r}\right\}^{2n}\frac{\{\Gamma(n+\frac{5}{2})\}^2}{(4n+1)!}$$

$$\times{}_3F_2\left(n+\frac{5}{2},\ n+\frac{5}{2},\ \frac{m}{2}+1;\ 1,\ \frac{m}{2}+2,\ \frac{r_0^2(\tau)}{r^2}\right),\ r>r_0(t) \qquad \ldots(4.16)$$

For $m=0$, the ${}_3F_2$ series of (4.16) reduces to a ${}_2F_1$ series. The term $\zeta_{\nu=0}$ corresponds to the wave elevation in a non-viscous fluid and is given by Sen (1963).

(ii) The expression for the surface displacement due to a more general pressure distribution of the form

$$F(r, t) = \sum_{m=0}^{\infty} \lambda_m(t) r^m$$

is as usual obtained by superposition of expressions (4.16) for $m=0$ to ∞.

(iii) Let $F(r, t)=\mathrm{D}'r^m t^{p'}$, $r_0(t)=(k't)^{1/2}$

where k' is a constant, and we suppose $m>-\frac{3}{2}$, $p'+\frac{1}{2}m+2>0$, the latter restriction being necessary for the convergence of the τ-integral at $\tau=0$. A direct evaluation of the τ-integral of (4.16) gives for this case

$$\zeta=\zeta_{\nu=0}+\frac{16\mathrm{D}'\Gamma(p'+\frac{1}{2}m+2)\nu t^{p'+1}(k't)^{m/2+1}}{\pi(m+2)g\rho r^4}\sum_{0}^{\infty}\left(\frac{2gt^2}{r}\right)^{2n+1}\frac{\{\Gamma(n+\frac{5}{4})\}^2(4n+2)}{\Gamma(p'+\frac{m}{2}+5+4n)}$$

$$\times\ {}_4F_3\left(n+\frac{5}{2}, n+\frac{5}{2}, \frac{m}{2}+1, p'+\frac{m}{2}+2; 1, \frac{m}{2}+2, p'+\frac{m}{2}+5+4n; \frac{k't}{r^2}\right),$$

$$r>(k't)^{1/2}. \qquad \ldots(4.17)$$

A numerical computation of the series so obtained is not very simple and this will not be attempted here.

Instead, we propose to derive asymptotic representations of ζ which are expected to be of great help in studying the properties of the wave motion.

5. Approximations to ζ

5.1 *Asymptotic Representation of the Surface Displacement* ζ

For the pressure $F(r, t)$ inside the blast wave, and the radius $r_0(t)$ of the spreading circular zone on the surface, we assume the following distribution

$$\left.\begin{aligned} F(r,t)&=A'(t+t_0)^{-n'}F\left(\frac{r}{r_0(t)}\right),\ r<r_0(t)\\ &\quad n' \text{ non-integral and } >1\end{aligned}\right\} \qquad \ldots(5.1)$$

$$r_0(t)=ut$$

where t_0 is the time (measured from the moment of explosion) at which the shock front first reaches the water surface. If the explosive source is situated sufficiently near the water surface, such a model for $F(r, t)$ closely approximates the one represented

by the first term in the perturbation scheme adopted by Sakurai (Holt 1965) for the determination of the pressure field inside the blast. Also to be noted are following (Howarth 1953) :

(i) at high pressures, $R(t+t_0)$ = radius of the shock wave $\propto (t+t_0)^{2/5}$,

$$\frac{dR}{dt} = \text{velocity of propagation of the shock} \propto (t+t_0)^{-3/5}$$

(ii) $r_0(t)=[R^2(t+t_0)-R^2(t_0)]^{1/2}$;

for convenience of analysis, we take $r_0(t)=ut$, (u=constant).

Using (5.1) into (3.1), we get

$$-g^{1/2}\rho\zeta/A' = \int_0^1 \alpha f(\alpha)d\alpha \int_0^\infty k^{3/2}J_0(kr)dk$$

$$\times \int_0^t \frac{r_0^3(\tau)\sin\sigma(t-\tau)e^{-2\nu k^2(t-\tau)}}{(t_0+\tau)^{n'}} J_0(k\alpha r_0(\tau))d\tau \qquad \ldots (5.2)$$

We seek to find the asymptotic behaviour of ζ under the conditions (which arise in course of the process of finding):

$$r>>r_0(t)>r_0(t_0),\ \frac{g(t+t_0)^2}{4r}>>1, \qquad \ldots (5.3)$$

it being supposed that the ratio t_0/t is not negligibly small.

Our method will consist in (i) replacing the Bessel function $J_0(k\alpha r_0(\tau))$ by its integral representation and interchange the order of integration with those of the integration with respect to τ and k, (ii) evaluating the τ-integral by introducing a function $F_n(a;\ t;\ t_0)$, (iii) approximating the k-integral by the method of stationary phase and by using the asymptotic value of F_n, and finally (iv) completing the θ-integration.

In carrying out (i), (5.2) gets replaced by

$$-\pi\rho g^{1/2}\zeta/(2A') = u^2\int_0^1 \alpha f(\alpha)d\alpha \int_0^{\pi/2} d\theta \int_0^\infty k^{3/2}J_0(kr)dk \times$$

$$\times \int_0^t \tau^2 (\tau+t_0)^{-n'} \cos(k\alpha u\tau \sin\theta)\sin\sigma(t-\tau)\, e^{-2\nu k^2(t-\tau)}\, d\tau \qquad \ldots (5.4)$$

Now

$$e^{-2\nu k^2(t-\tau)} \cos(k\alpha u\tau \sin\theta) \sin\sigma(t-\tau)$$
$$= \tfrac{1}{2} e^{-2\nu k^2 t} \operatorname{Im}\left[e^{i\sigma t}\left\{ e^{\tau(2\nu k^2 + i(ku\alpha \sin\theta - \sigma))} + e^{\tau(2\nu k^2 - i(ku\alpha \sin\theta + \sigma))}\right\}\right]$$

We use this in (5.4) and complete the τ-integration in terms of the function $F_n'(a; t; t_0)$ introduced as follows :

$$e^{-at_0} F_n'(a; t; t_0) = \int_0^t \tau^2 (\tau + t_0)^{-n'} e^{at} d\tau, \quad \operatorname{Re} a \geqslant 0$$
$$= e^{-at_0} \int_{t_0}^{t+t_0} (\tau^{2-n'} - 2t_0\tau^{1-n'} + t_0^2\tau^{-n'}) e^{a\tau} d\tau$$
$$= e^{-at_0}\left[\frac{\tau^{3-n'}}{3-n'} {}_1F_1(3-n'; 4-n'; a\tau) - 2t_0 \frac{\tau^{2-n'}}{2-n'} {}_1F_1(2-n'; 3-n'; a\tau) + t_0^2 \frac{\tau^{1-n'}}{1-n'} {}_1F_1(1-n'; 2-n'; a\tau)\right]_{\tau=t_0}^{\tau=t+t_0} \quad \ldots(5.5)$$

where a stands for either of the expressions

$$a_j = 2\nu k^2 - i\{\sigma + (-1)^j ku\alpha \sin\theta\}, \; j = 1, 2. \quad \ldots(5.6)$$

Here ${}_1F_1$ denotes Kummer's confluent hypergeometric function.

It is known that when $kr >> 1$,

$$J_0(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} \cos\left(kr - \frac{\pi}{4}\right).$$

This result is used together with (5.5) in (5.4) to reduce the k-integral of (5.4) to the following form :

$$(8\pi r)^{-1/2} \operatorname{Im} \int_0^\infty k e^{-2\nu k^2(t+t_0)} \sum_{j=1}^{2}{}' F_n'(a_j(k); t; t_0) \times$$
$$\times \left\{ e^{ir\{\sigma(t+t_0)r^{-1} + (-1)^j r^{-1} ku\alpha t_0 \sin\theta - k + \frac{\pi}{4r}\}} + e^{ir\{\sigma(t+t_0)r^{-1} + (-1)^j r^{-1} ku\alpha \sin\theta + k - \frac{\pi}{4r}\}}\right\} dk. \quad \ldots(5.7)$$

The phase function

$$\sigma(t+t_0)r^{-1}+(-1)^j r^{-1}ku\alpha t_0 \sin\theta+k-\frac{\pi}{4r}$$

has no stationary point in $0<k<\infty$ for $j=2$, and since $ut_0<<r$, the same statement is true for $j=1$. Therefore the contribution of the part of the k-integral containing the second exponential in the expression (5.7) is $O(r^{-1})$ as $r\to\infty$.

If the other phase function is designated by $P(k)$, the stationary point $k = k_0$ is given by $P'(k) = 0$

where

$$k_0 = \frac{g(t+t_0)^2}{4r^2\{1-(-1)^j \alpha u t_0 r^{-1}\sin\theta\}^2} \qquad \ldots (5.8)$$

The principle of stationary phase enables us to replace (5.7) by the following :

The k-integral of (5.4) $= 2^{-1/2} r^{-1}\{1-(-1)^j r^{-1}\alpha u t_0 \sin\theta\}^{-1/2} k_0^{3/2} e^{-2\nu k_0^2(t+t_0)}$

$$\times \text{Im} \sum_{j=1}^{2} F_n'\left(a_j(k_0); t; t_0\right) \exp\left[\frac{ig(t+t_0)^2}{4r}\left\{1+(-1)^j \alpha u t_0 r^{-1}\sin\theta\right\}\right]+O(r^{-3/2}),$$

$$\text{as } r\to\infty, \quad \frac{g(t+t_0)^2}{4r} >> 1. \qquad \ldots (5.9)$$

To simplify (5.9), we may then use the asymptotic development of the function ${}_1F_1$ of (5.5) for large arguments [Erdelyi 1953, 6.13.1 (2)] and thus obtain the approximate value of the function $F_n'(a; t; t_0)$:

$$F_n'(a; t; t_0) \sim \frac{t^2}{a}(t+t_0)^{-n'} e^{a(t+t_0)} \qquad \ldots (5.10)$$

Keeping in mind the restrictions mentioned in (5.3), we now calculate the following :

$$[a_j(k_0)]^{-1} \simeq \left[2\nu k_0^2 - \frac{ig(t+t_0)}{2r}\left\{1+(-1)^j \frac{\alpha u t}{2r}\sin\theta\right\}\right]^{-1}$$

$$\simeq \left[2\nu k_0^2 - \frac{ig(t+t_0)}{2r}\right]^{-1}$$

$$a_j(k_0)(t+t_0) = 2\nu k_0^2 (t+t_0) - i\left[\frac{g(t+t_0)^2}{2r}\left\{1+(-1)^j \frac{\alpha u t_0}{r}\sin\theta\right\}\right.$$

$$\left. + (-1)^j g(t+t_0)^3 \frac{\alpha u \sin\theta}{4r^2}\right].$$

Using these results together with (5.10) in (5.9), we get after an approximation in the exponential factor there by Taylors' expansion,

the k-integral of (5.4)

$$\simeq \frac{g^{1/2}t^2\,(t+t_0)^{2-n'}}{2^{3/2}\,r^3}\left[\cos\frac{g\,(t+t_0)^2}{4r}\cos\left(\frac{g\,(t+t_0)^2}{4r}\cdot\frac{\alpha ut}{r}\sin\theta\right)\right.$$
$$\left.-\frac{\nu g\,(t+t_0)^3}{4r^3}\sin\frac{g(t+t_0)^2}{4r}\cos\left(\frac{g\,(t+t_0)^2}{4r}\cdot\frac{\alpha ut}{r}\sin\theta\right)\right] \quad \ldots\ (5.11)$$

Introducing this approximation in (5.4) and using again the integral representation for $J_0\,(z)$, we complete the θ-integration to obtain, under the restrictions mentioned in (5.3).

$$\zeta = -\frac{A'u^2t^2\,(t+t_0)^{2-n'}}{2^{3/2}\,r^3\rho}\left[\cos\frac{g\,(t+t_0)^2}{4r}-\frac{\nu g\,(t+t_0)^3}{4r^3}\sin\frac{g(t+t_0)^2}{4r}\right]$$
$$\times\,\bar{f}\left(\frac{g(t+t_0)^2}{4r}\cdot\frac{ut}{r}\right) \quad \ldots\ (5.12)$$

where

$$\bar{f}(k) = \int_0^1 \alpha\, f(\alpha)\, J_0\,(k\alpha)\, d\alpha. \quad \ldots\ (5.13)$$

The term containing ν in (5.12) thus measures the correction to the wave elevation due to a small viscosity in the liquid.

5.2 *A Particular Case*

When

$$f\left(\frac{r}{r_0(t)}\right) = \frac{D}{A'}\ , \qquad (D = \text{constant})$$

$$\bar{f}\,(k) = \frac{D}{A'}\,J_1\,(k)/k. \quad \ldots\ (5.14)$$

Equation (5.12) then becomes

$$\zeta = -\frac{D}{g\rho}\cdot\frac{2^{1/2}ut}{r}\,(t+t_0)^{-n'}\,J_1\left(\frac{g(t+t_0)^2\,ut}{4r^2}\right)$$
$$\times\left[\cos\frac{g\,(t+t_0)^2}{4r}-\nu\,\frac{g\,(t+t_0)^3}{4r^3}\sin\frac{g\,(t+t_0)^2}{4r}\right] \quad \ldots\ (5.15)$$

For $\nu = 0$, this recovers the formula obtained by Sen (1963).

For $t_0 = 0.5$ sec., $u = 0.05$ m/sec., $\nu = .0109$, $n' = \frac{3}{4}$ the variation of $\zeta^{(\nu)}$—the viscosity-affected part of ζ with t at a distance $r = 10$ m is illustrated in Fig. 1 and the variation of $\zeta^{(\nu)}$ with r for $t = 9.5$ sec. is illustrated in Fig. 2.

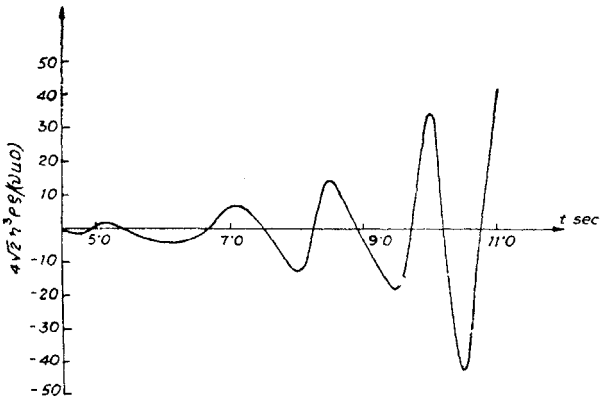

FIG. 1.

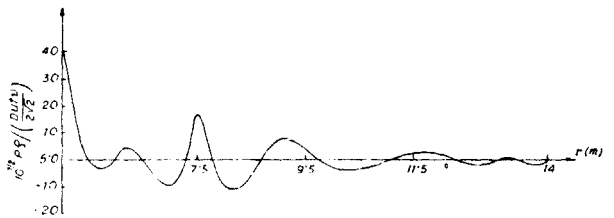

FIG. 2.

5.3 *Case of a Weightless Fluid*

In view of the fact that the pressure effects are more marked than the gravity influence in the immediate area of the blast, it seems appropriate here to investigate the motion of the a weightless fluid.

If we put $g = 0$ in (4.14), we get

$$(\zeta)_{g=0} = (\zeta)_{\substack{\nu=0\\g=0}} + 18\,\nu\rho^{-1}\, r^{-5}\int_0^1 \alpha\, d\alpha \int_0^t r_0^2(\tau)\, F\Big(\alpha r_0(\tau), \tau\Big)(t-\tau)^2 \times$$

$$\times\, {}_2F_1\left(\tfrac{5}{2}, \tfrac{5}{2}; 1; \frac{\alpha^2 r_0^2(\tau)}{r^2}\right) d\tau, \quad (r > r_0(t)) \quad \ldots \quad (5.16)$$

The term $(\zeta)_{\substack{\nu=0\\g=0}}$ is the value of ζ in a non-viscous fluid and is found from Sen (1963).

A Particular Case

$$\text{Let } F(r,t) = D'\, r^m t^{p'}, \ (m > -\tfrac{3}{2},\ m+p'+3>0),\ r_0(t) = ut. \qquad \ldots (5.17)$$

Then from (4.16)

$$(\zeta)_{g=0} = (\zeta)_{\substack{\nu=0\\g=0}} + \nu\, \frac{9\, D'\, t^{p'+3}}{\rho\,(m+2) r^5}\, (ut)^{m+2} \Big[\, B(1, \tfrac{1}{2}p' + \tfrac{1}{2}m + \tfrac{3}{2}) \times$$

$$\times {}_4F_3\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}m+1, \tfrac{1}{2}p'+\tfrac{1}{2}m+\tfrac{3}{2}; 1, \tfrac{1}{2}m+2, \tfrac{1}{2}p'+\tfrac{1}{2}m+\tfrac{5}{2}; \frac{u^2t^2}{r^2}\right)$$

$$-2B\left(1, \tfrac{1}{2}p'+\tfrac{1}{2}m+2\right) {}_4F_3\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}m+1, \tfrac{1}{2}p'+\tfrac{1}{2}m+2; 1, \tfrac{1}{2}m+2, \tfrac{1}{2}p'+\tfrac{1}{2}m+3; \frac{u^2t^2}{r^2}\right)$$

$$+B\left(1, \tfrac{1}{2}p'+\tfrac{1}{2}m+\tfrac{5}{2}\right) {}_4F_3\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}m+1, \tfrac{1}{2}p'+\tfrac{1}{2}m+\tfrac{5}{2}; 1, \tfrac{1}{2}m+2, \tfrac{1}{2}p'+\tfrac{1}{2}m+\tfrac{7}{2}; \frac{u^2t^2}{r^2}\right)\Big]. \quad \ldots(5.18)$$

The term $(\zeta)_{\substack{\nu=0\\ g=0}}$ is the value of ζ in a non-viscous fluid and is given by Sen (1963).

5.4 *Some Characteristics of the Motion as Affected by Viscosity*

We shall base our observations mainly on the part $(\zeta-\zeta_{\nu=0})$ as given by (5.12) and (5.15) naming these as $\zeta_\nu^{(1)}$ and $\zeta_\nu^{(2)}$ respectively.

As is well-known, the common argument of the trigonometric factors in (5.12) may be interpreted as the phase, and the coefficients of these factors are the respective amplitudes of the component wave-motions represented by $(\zeta)_{\nu=0}$ and $\zeta_\nu^{(1)}$. It is noted at once that viscosity does not affect the phase of either motion, and it is only the amplitude of the last-named motion which is being modified in the process.

The ratio of the amplitudes of $\zeta_\nu^{(1)}$ and $\zeta_{\nu=0}$ is $\dfrac{\nu g(t+t_0)^3}{r^3}$ in absolute value.

For $t>>t_0$, the amplitude of $\zeta_\nu^{(1)}$ varies as

$$\frac{t^{7-n'}}{r^6}\bar{f}\left(\frac{gut^3}{4r^2}\right).$$

The times of maximum amplitude corresponding to $\zeta_\nu^{(1)}$ at any point is given by

$$t_m=\left(\frac{4r^2a'_m}{ug}\right)^{1/3}$$

where a'_m is the mth root of the equation

$$(7-n')\bar{f}(x)+3x\bar{f}'(x)=0.$$

For $\zeta_\nu^{(2)}$, this equation becomes

$$3xJ_0(x)=(n'-1)J_1(x).$$

Therefore the points of maximum amplitude, corresponding to $\zeta_\nu^{(1)}$ or $\zeta_\nu^{(2)}$ at a distance r move outwards with corresponding velocities

$$\left(\frac{27gur}{32a'_m}\right)^{1/3}.$$

At times $T_m=\left(\frac{4r^2b_m}{gu}\right)^{1/3}$, where b_m is the mth root of the equation $J_1(x)=0$, the amplitude of the whole motion becomes zero. The points of minimum amplitude at a distance r move outwards with the corresponding velocities $\left(\frac{27gur}{32b_m}\right)^{1/3}$. Again, when $\left(\frac{gut^3}{4r^2}\right)>>1$, the asymptotic expansion of $J_1\left(\frac{gut^3}{4r^2}\right)$ shows that the amplitude, corresponding to $\zeta_\nu^{(2)}$, is almost of the order of $t^{5/2-n'}r^{-3}$.

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