STABILITY PROBLEMS OF PERTURBED DIFFERENCE EQUATIONS WITH RETARDED ARGUMENT

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This paper is concerned with the stability, boundedness and asymptotic behaviour of solutions of perturbed difference equations with retarded argument. These results extend known results from ordinary difference equations to difference equations with retarded argument and differs from them in a number of significant respects.

1. INTRODUCTION

The theory of difference equations is in a process of continuous development and it has become significant for its various applications. Recently, Coffman (1964), Freeman (1965), Hurt (1967), Maslovskaya (1966), Smith (1966), Sugiyama (1969) and others have dealt with the difference equations and obtained several remarkable results. It seems, however, that, if we are concerned with various results in the theory of difference equations with retarded argument, not so many papers have appeared so far, although, these results are of great importance both in theory and applications. We, in the present paper, wish to investigate these problems further. We first establish a basic finite difference inequality in the theory of difference equations with retarded argument. Then we use this inequality to study the stability, boundedness and asymptotic behaviour of solutions of perturbed difference equations with retarded argument. It is also expected that some results obtained here can be applied to the error estimates in the numerical analysis. In particular, our results are motivated by two recent studies of Coffman (1964) and Sugiyama (1969) dealing with asymptotic behaviour of solutions of ordinary difference equations and stability problems of difference and functional difference equations.

2. NOTATIONS AND BASIC THEOREM

We begin by stating some notations which will be used in our subsequent discussion. Let N denote the set of integers $n, n \geqslant n_0$ where $n_0 \geqslant 0$ is an integer and R^r, the r-dimensional vector space. For any $x \in R^r$, $\| x \|$ denote any convenient vector norm in R^r. We define the operator ∇ such that $\nabla x(n) = x(n) - x(n-1)$, for $n \in N$.

The following basic theorem on finite difference inequalities is useful for the applications we have in view.

Theorem 1—Let the real valued function $g(n, r)$ be defined for $n \in N$, $0 \leqslant r < \infty$, and monotone increasing with respect to r for any fixed $n \in N$. If there exist real valued functions $u(n)$ and $v(n)$ which satisfy the inequalities

$$\nabla u(n) \leqslant g(n, u(n-1)) \text{ and } \nabla v(n) \geqslant g(n, v(n-1)) \qquad ...(2.1)$$

for all integers $n \geqslant n_0$ and

$$u(n_0-1) \leqslant v(n_0-1)$$

then there holds an inequality

$$u(n) \leqslant v(n) \qquad ...(2.2)$$

for all integers $n \geqslant n_0$.

PROOF : We assume that the inequality $u(n) \leqslant v(n)$ is not satisfied for all $n \geqslant n_0$. Then there exists an integer β such that $u(n) \leqslant v(n)$ for all $n \in [n_0, \beta-1]$, but

$$u(\beta) > v(\beta). \qquad ...(2.3)$$

Then from (2.1) and (2.3) we obtain

$$v(\beta-1) + g(\beta, v(\beta-1)) \leqslant v(\beta) < u(\beta) \leqslant u(\beta-1) + g(\beta, u(\beta-1)).$$

This is a contradiction is view of the monotone property of $g(n, r)$.
and hence the result follows.

Remark 1 : If in Theorem 1, the inequalities in (2.1) be replaced by the inequality

$$\nabla u(n) - g(n, u(n-1)) \leqslant \nabla v(n) - g(n, v(n-1))$$

for each $n \in N$, then, the conclusion (2.2) of theorem 1 remains valid.

Remark 2 : Theorem 1 can be used in seeking bounds for solutions of difference equations with retarded argument. For example, if $x(n)$ is a solution of difference equation

$$\nabla x(n) = g(n, x(n-1)), \; n \geqslant n_0 \qquad ...(2.4)$$

under the initial condition $x(n_0-1) = x_0$, where g is the same function as defined in Theorem 1. If there exist real valued functions $u(n)$ and $v(n)$ which satisfy the inequalities in (2.1), then by repeated applications of Theorem 1, we conclude that

$$u(n) \leqslant x(n) \leqslant v(n), \; n \in N.$$

3. Perturbed Difference Systems

We consider the perturbed difference system

$$\nabla z(n) = A(n)\, z(n-1) + f(n, z(n-1)),\ n \geqslant n_0 \qquad \ldots(3.1)$$

under the initial condition $z(n_0-1) = z_0$, where the function $f(n, z)$ is defined for $n \geqslant n_0$ and $\| z \| < H\ (H>0)$, $f(n, 0)=0$ for any $n \in N$, and $A(n)$ is a $r \times r$ matrix with $\det A(n) \neq 0$ for any $n \geqslant n_0$.

If $y_1(n), y_2(n), \ldots, y_r(n)$ are linearly independent solutions of

$$\nabla\, y(n) = A(n)\, y(n-1),\ n \geqslant n_0 \qquad \ldots(3.2)$$

with the initial condition $y(n_0-1)=z_0$, then $Y(n)=(y_1(n), \ldots\ y_r(n))$ is called a fundamental matrix of (3.2). If the initial condition $Y(n_0-1)=I$ (the unit matrix) is imposed, then we obtain $Y(n, s)= Y(n)\, Y^{-1}(s)$ (see Freeman 1965). It is well known that there exists a fundamental matrix if $\det A(n) \neq 0$. It is noted that $\| Y \|$ represents the natural norm of the fundamental matrix Y which is associated with the norm used for vectors.

Definition 1—We say that a linear system (3.2) is exponentially asymptotically stable if there exist positive constants K and α such that

$$\| Y(n)\ Y^{-1}(s) \| \leqslant K e^{-\alpha(n-s)},\quad n \geqslant s \geqslant 0$$

where $Y(n)$ is a fundamental matrix of (3.2) as defined above.

There are many results which relate stability, boundedness and asymptotic behaviour of the trivial solutions of (3.1) to that of the trivial solutions of (3.2) without retarded argument. References to some of the work on such systems without retarded argument can be found in the bibliographies of Coffman (1964), Freeman (1965), Maslovskaya (1966) and other readily available literature.

During the last few years a number of investigators have studied the perturbed difference systems of the type (3.1) by comparing its solution with the solutions of the linear system of the type (3.2) without retarded argument, when the perturbation term $f(n, z)$ satisfies the condition $\| f(n, z) \| \leqslant c \| z \|$ for sufficiently small constant $c > 0$. We prove the following theorem which is useful to discuss the stability and boundedness for cases when $f(n, z)$ in (3.1) is not necessarily small.

Theorem 2—Let $Y(n)$ be the fundamental matrix of (3.2) such that $Y(n_0-1)=I$. Let $\| Y(n) \| \leqslant \alpha(n)$, where $\alpha(n)$ is a positive real valued function defined for $n \in N$ and let $\alpha(n_0-1) = \alpha_0$. Suppose further that the function $f(n, z)$ in (3.1) satisfies the inequality

$$\| Y^{-1}(n) f(n, z(n-1)) \| \leqslant g\left(n, \frac{1}{\alpha(n-1)} \| z(n-1) \| \right),\ n \geqslant n_0 \qquad \ldots(3.3)$$

where the function $g(n, u)$ is defined for $n \in N$, $0 \leqslant r < \infty$, and monotone increasing with respect to u for any fixed $n \in N$. Let $u(n)$ be any solution of the difference equation

$$\nabla u(n) = g(n, u(n-1,)) \quad n \geqslant n_0 \qquad \text{...(3.4)}$$

with the initial condition $u(n_0 - 1) = u_0$. Then

$$\| z(n) \| \leqslant \alpha(n)\, u(n,) \quad n \geqslant n_0 \qquad (3.5)$$

where $z(n)$ is any solution of (3.1) such that $\| z_0 \| \leqslant \alpha_0 u_0$.

PROOF : Using the variation of parameters formula, any solution of (3.1) is represented by

$$z(n) = Y(n) z_0 + Y(n) \sum_{s=n_0}^{n} Y^{-1}(s) f(s, z(s-1)), \quad n \geqslant n_0.$$

Then, by using the given hypotheses we have

$$\frac{1}{\alpha(n)} \| z(n) \| \leqslant \| z_0 \| + \sum_{s=n_0}^{n} g\left(s, \frac{1}{\alpha(s-1)} \| z(s-1) \| \right), \quad n \geqslant n_0. \qquad \text{...(3.6)}$$

Let the right-hand side of (3.6) be denoted by $m(n)$. Then we obtain

$$\nabla m(n) = g\left(n, \frac{1}{\alpha(n-1)} \| z(n-1) \| \right), \quad n \geqslant n_0.$$

Since $\frac{1}{\alpha(n)} \| z(n) \| \leqslant m(n)$, for $n \geqslant n_0$ and $g(n, u)$ is monotone increasing with respect to u, it follows that

$$\nabla m(n) \leqslant g(n, m(n-1)), \quad n \geqslant n_0. \qquad \text{...(3.7)}$$

Applying Theorem 1 to (3.7) and (3.4), we obtain the desired result (3.5).

We note that the inequality (3.5) implies the boundedness of $\| z(n) \|$ if $\alpha(n) u(n)$ is bounded.

Remark 3 : Theorem 2 involves comparison between solutions of the system (3.1) and the scalar difference equation (3.4). Assuming that the scalar difference equation (3.4) is stable in some sense, one can prove that the system (3.1) has a corresponding stability property.

It is possible to construct an example of difference equation of the form (3.4) having the desired stability behaviour.

Example 1—Consider the scalar difference equation

$$\nabla u(n) = (e^{-1}-1)u(n-1) \qquad \ldots(3.8)$$

defined for $n \geqslant n_0$, with the given initial condition $u(n_0-1) = u_0$. It is easy to observe that the solution

$$u(n) = u_0 \, e^{-(n+1-n_0)}$$

of (3.8) is asymptotically stable.

We now state and prove the following asymptotic stability theorem for the system (3.1).

Theorem 3—Let the function $f(n, z)$ in (3.1) satisfy an inequality

$$\|Y^{-1}(n) f(n, Y(n-1)x(n-1))\| \leqslant g(n, \|x(n-1)\|), \; n \geqslant n_0 \qquad \ldots(3.9)$$

where $Y(n)$ is a fundamental matix of (3.2) such that $Y(n_0-1)=I$, and $g(n, u)$ is the same function as defined in Theorem 2. Suppose that any solution of (3.4) is bounded. Then, if the trivial solution of (3.2) is exponentially asymptotically stable, the trivial solution of (3.1) is also exponentially asymptotically stable.

PROOF : Let $z(n)$ be any solution of (3.1). The substitution $z(n) = Y(n)x(n)$ transforms (3.1) to the system

$$\nabla x(n) = Y^{-1}(n) f(n, Y(n-1)x(n-1)), \; n \geqslant n_0$$

which in view of (3.9) reduces to

$$\nabla \|x(n)\| \leqslant \|Y^{-1}(n) f(n, Y^{-1}(n-1)x(n-1))\| \leqslant g(n, \|x(n-1)\|). \qquad \ldots(3.10)$$

Applying Theorem 1 to (3.10) and (3.4) we have $\|x(n)\| \leqslant u(n), n \geqslant n_0$, provided $\|x(n_0-1)\| \leqslant u_0$. Hence, it follows from Definition 1 that

$$\|z(n)\| \leqslant \|Y(n)\| . \| x(n)\| \leqslant K \exp(-\alpha(n-n_0+1)) \, u(n), \; n \geqslant n_0$$

which implies the desired result, since $u(n)$ is bounded.

Perturbation problems, which consider the inference of asymptotic properties of the solutions of a differential equation, have been the subject of many current research papers in ordinary differential equations. In the following theorem we investigate the asymptotic behaviour of solutions of the nonlinear perturbed difference system (3.1) generated by the solutions of the corresponding linear system (3.2).

The following lemma will be needed in the proof of our next theorem.

Lemma 1—Let the function $f(n, z)$ in (3.1) satisfy an inequality

$$\| Y^{-1}(n) f(n, Y(n-1)x(n-1)) \| \leqslant h(n) . \| x(n-1) \|, \; n \geqslant n_0 \qquad \ldots(3.11)$$

where $Y(n)$ is a fundamental matrix of (3.2) such that $Y(n_0-1) = I$, $h(n)$ is defined for $n \in N$ and $\sum_{s=n_0}^{\infty} h(s) < \infty$. Then for any solution $z(n)$ of (3.1), the function $Y^{-1}(n)\, z(n)$ has a finite limit as $n \to \infty$.

PROOF : If we apply the same transformation $z(n) = Y(n)x(n)$, then the difference system (3.1) reduces to

$$\nabla x(n) = Y^{-1}(n)\, f(n,\ Y(n-1)x(n-1)),\ n \geqslant n_0. \qquad \text{(3.12)}$$

Since, for any integers $n_1, n_2 > n_0$ $(n_1 \leqslant n_2)$, we have from (3.12) and (3.11)

$$\|x(n_2)-x(n_1)\| \leqslant \sum_{s=n_1}^{n_2} \| Y^{-1}(s)\, f(s,\ Y(s-1)\, x(s-1)) \|$$

$$\leqslant \sum_{s=n_1}^{n_2} h(s)\, \| x(s-1) \|, \quad n \geqslant n_0$$

by which we can easily obtain our result.

We are now prepared to state and prove the following theorem.

Theorem 4—Suppose that $A(n)$ is a $r \times r$ matrix defined for $n \in N$, $\det A(n) > 0$, $\prod_{s=n_0}^{\infty} \det A(s) > 0$, and all solutions of (3.2) are bounded as $n \to \infty$. Suppose further that the function $f(n, z)$ in (3.1) satisfies an inequality

$$\| f(n,\ z(n-1)) \| \leqslant h(n)\, \| z(n-1) \|, \quad n \geqslant n_0$$

where $h(n)$ is defined for $n \in N$ and $\sum_{s=n_0}^{\infty} h(s) < \infty$. Then for any solution $z(n)$ of (3.1), there exists a solution $y(n)$ of (3.2) such that

$$\lim_{n \to \infty} (z(n) - y(n)) = 0. \qquad \text{...(3.13}$$

PROOF : Let $z(n)$ be any solution of (3.1). The substitutions $z(n) = Y(n)\, x(n)$ transforms (3.1) to the system

$$\nabla x(n) = Y^{-1}(n) f(n, Y(n-1)x(n-1)), \quad n \geqslant n_0.$$

Then by using the given hypotheses we have

$$\| Y^{-1}(n) f(n, Y(n-1)x(n-1)) \| \leqslant \| Y^{-1}(n) \| . \| Y(n-1) \| . h(n) . \| x(n-1) \|$$
$$\leqslant B\, h(n) \| x(n-1) \|, \quad n \geqslant n_0$$

where B is constant. An application of Lemma 1 yields the existence of a limit c of $x(n)$ as $n \to \infty$. Since a function $y(n) = Y(n)\, c$ is a solution of (3.2) with an initial condition $y(n_0 - 1) = c$. To verify that (3.13) holds we observe that

$$\lim_{n \to \infty} (z(n) - y(n)) = \lim_{n \to \infty} Y(n)\, (x(n) - c) = 0.$$

This completes the proof of the theorem.

Finally we note that there is no essential difficulty in obtaining various results concerning the other problems in the theory of difference equations with retarded argument, for example, the problems of total stability, almost periodicity, asymptotic manifold, and so on. Since this translation is quite straightforward in view of the results of this paper and requires no fresh insight, we do not discuss it here.

There are many possible additional topics we have not touched upon, as, e.g., difference equations with retarded argument on the set of all integers, or dependence on A [for a given operator-valued function $A(n)$ in equation (3.1)] of the properties of the solutions. It might turn out to be interesting to consider also difference equations with a real variable and a continuity assumption on the solutions; they may help to approach the much more essential topic of functional-differential equations.

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