ON THE BOUNDEDNESS OF SOLUTIONS OF DIFFERENCE-DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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(Communicated by B. R. Seth, F.N.A.)

(*Received* 18 *December* 1972)

The object of the present paper is to establish some theorems on the boundedness of solutions of difference-differential equations in Hilbert space which are, in general, not linear.

§1. Difference-differential equations, or functional-differential equations as they are frequently called, have become important in many fields. These equations have been studied by many authors and the most comprehensive work on this topic is a book by Bellman and Cooke (1963). In an interesting paper, Schaeffer (1968) has established a boundedness theorem for the solutions of linear differential equations in Hilbert space. Recently, Kartsatos (1969) has proved a boundedness theorem for the solutions of nonlinear differential equations in Hilbert space, which contains partially as a special case a result of Schaeffer (1968). It is the purpose of this paper to establish some results concerning the boundedness of solutions of difference-differential equations in Hilbert space, which extends the results of Bellman and Cooke (1963), Kartsatos (1969) and Sugiyama (1960).

§2. Let I denote the interval $0 \leqslant t < \infty$, $H = (H, \langle \cdot, \cdot \rangle)$ denote a complex Hilbert space, and $B = B(H, H)$, the space of all bounded linear operators from H into H, associated with the strong operator topology. The only topology that we consider on H is the strong one. For any $x \in H$, $\| x \|$ denote any convenient norm in H.

We first consider below the difference-differential equation

$$\dot{x}(t) = A(t)\, x(t) + f(t, x(t), x(t-1)), \quad 0 \leqslant t < \infty. \qquad \ldots(2.1)$$

under the initial conditions

$$x(t-1) = \phi(t) \ (0 \leqslant t < 1) \text{ and } x(0) = x_0 \qquad \ldots(2.2)$$

where $\phi(t)$ is continuous for $0 \leqslant t < 1$, and $\lim_{t \to 1-0} \phi(t) = \phi(1-0)$ exists, $x : I \to H$, is a differentiable function on I with continuous first derivative, $A : I \to B$ is a continuous function on I, and $f : I \times H \times H \to H$ is also continuous on $I \times H \times H$. It is supposed that the existence of solutions of (2.1) with (2.2) is guaranteed for $0 \leqslant t < \infty$.

We shall now establish the following boundedness theorem for the solutions of (2.1) under the initial conditions (2.2).

Theorem 1—Consider (2.1) with (2.2) under the following assumptions :

(i) There exists an operator valued function $Q : I \to B$ continuous and such that

$$\dot{Q}(t)+Q(t)\,A(t)+A^*(t)\,Q(t)=0,\ t\in I$$

and

(ii) $g(\|x\|) \leqslant |<Q(t)\,x, x>|,\ (t, x)\in I\times H$

where $A^*(t)$ is the adjoint of the operator $A(t)$, $g: R_+\to R_+=[0, +\infty)$ is continuous and $\lim_{r\to+\infty} (g)r=+\infty$;

(iii) $\|x\|.\|f(t, x, y)\|\leqslant h(t)\,(g(\|x\|)+g(\|y\|))$

where $h : I\to R_+$ is a continuous function and such that

$$\int_0^\infty h(t)\,\|Q(t)\|\,dt<+\infty$$

Then, any solution of (2.1) with (2.2) is bounded for $0\leqslant t<\infty$.

PROOF : Let $x(t)$ be a solution of (2.1) with (2.2). By differentiation of the function

$$V(t)=<Q(t)x(t), x(t)>$$

we have

$$\begin{aligned}\dot{V}(t)&=<\dot{Q}(t)x(t)+Q(t)\,\dot{x}(t), x(t)>+<Q(t)x(t),\ \dot{x}(t)>\\ &=<\dot{Q}(t)x(t)+Q(t)A(t)x(t)+Q(t)f(t, x(t),x(t-1)), x(t)>\\ &\quad+<Q(t)x(t), A(t)x(t)+f(t, x(t), x(t-1))>\\ &=<(\dot{Q}(t)+Q(t)A(t)+A^*(t)Q(t))x(t), x(t)>\\ &\quad+<Q(t)\,f(t, x(t), x(t-1)), x(t)>+<Q(t)x(t), f(t, x(t), x(t-1))>\end{aligned}$$

and by integration from 0 to t $(0\leqslant t)$ we have

$$\begin{aligned}V(t)=V(0)+\int_0^t [&<Q(s)f(s, x(s), x(s-1)), x(s)>\\ &+<Q(s)x(s), f(s, x(s), x(s-1))>]ds. \qquad ...(2.3)\end{aligned}$$

Now, we have to consider two cases.

Case I : $0 \leqslant t \leqslant 1$—It follows from (2.2), (ii), (iii), (2.3) that

$$g(\|x(t)\|) \leqslant |V(t)| \leqslant |V(0)| + 2\int_0^t \|Q(s)\|.\|f(s, x(s,) \phi(s))\|.\|x(s)\|ds$$

$$\leqslant |V(0)| + 2\int_0^t \|Q(s)\|.h(s) \, (g(\|x(s)\|) + g(\|\phi(s)\|))ds$$

$$\leqslant c + 2\int_0^t \|Q(s)\|.h(s) \, g(\|x(s)\|)ds$$

where

$$c = |V(0)| + 2\int_0^1 \|Q(s)\|.h(s) g(\|\phi(s)\|)ds.$$

By Gronwall's lemma (Bellman and Cooke 1963), the above inequality yields the estimation

$$g(\|x(t)\|) \leqslant c \exp\left(2\int_0^t h(s)\|Q(s)\| \, ds\right) \leqslant c \exp\left(2\int_0^\infty h(s)\|Q(s)\|ds\right). \quad \ldots(2.4)$$

Case II : $1 \leqslant t < \infty$—It follows from (2.2), (ii), (iii), (2.3) that

$$g(\|x(t)\|) \leqslant |V(t)| \leqslant |V(0)| + 2\int_0^1 \|Q(s)\|.\|f(s, x(s), \phi(s)\|.\|x(s)\|ds$$

$$+ 2\int_1^t \|Q(s)\|.\|f(s, x(s), x(s-1))\|.\|x(s)\|ds$$

$$\leqslant |V(0)| + 2\int_0^1 \|Q(s)\|.h(s)(g(\|x(s)\|) + g(\|\phi(s)\|))ds$$

$$+ 2\int_1^t \|Q(s)\|.h(s) \, (g(\|x(s)\|) + g(\|x(s-1)\|))ds$$

$$\leqslant c + 2\int_0^t \|Q(s)\|.(h(s) + h(s+1))g\,(\|x(s)\|)ds$$

where c is as defined above. Now an application of Gronwall's lemma, the above inequality yields the estimation

$$g(\|x(t)\|) \leqslant c \exp(2\int_0^t \|Q(s)\|.(h(s)+h(s+1))ds)$$

$$\leqslant c \exp(4\int_0^\infty \|Q(s)\|.h(s)ds) \quad \ldots(2.5)$$

which implies together with (2.4) the boundedness of $\| x(t)\|$.

§3. As for difference-differential equations of netural type, we shall establish a boundedness theorem, for which the equation to be discussed here is

$$\dot{x}(t) = A(t)x(t)+f(t, x(t), x(t-1), \dot{x}(t-1)), \; 0\leqslant t<\infty$$

under the initial conditions ...(3.1)

$$x(t-1) = \phi(t) \quad (0\leqslant t<1) \quad \text{and} \quad x(0) = x_0 \quad \ldots(3.2)$$

where $A(t)$ is as defined above, $f(t, x, y, z)$ is continuous and bounded, $\|f(t, x, y, z)\| \leqslant M$, for $0 \leqslant t<\infty$, $x, y, z \in H$, and $\phi(t)$, is a given function as before, continuously differentiable for $0\leqslant t<1$, $\lim_{t\to 1-0} \dot{\phi}(t)$, $\lim_{t\to+0} \dot{\phi}(t)$ exist. It is supposed that the existence of solutions of (3.1) with (3.2) is guaranteed for $0\leqslant t<\infty$.

Theorem 2—Consider (3.1) with (3.2) under the assumptions (i) and (ii) of Theorem 1. Further we suppose that the inequality

$$\|x\|.\|f(t, x, y, z)\|\leqslant h(t).(g(\|x\|)+g(\|y\|)+g(\|z\|))$$

is satisfied for $0\leqslant t<\infty$, where $h(t)$ is the same function as defined in Theorem 1 such that

$$\int_0^t h(t)\,\|Q(t)\|\,dt<+\infty.$$

Then, any solution of (3.1) with (3.2) is bounded for $0\leqslant t<\infty$.

The proof of this theorem is analogous to the proof of Theorem 1. We omit the details.

Theorems 1 and 2 may also be extended to the non-linear equations

$$\dot{x}(t)=A(t)\,x(t)+f(t, x(t), x(t-h))$$

and

$$\dot{x}(t)=A(t)\,x(t)+f(t, x(t), \dot{x}(t-h), x(t-h))$$

respectively, where $h \geqslant 0$ is a constant retardation and A, x, f are the functions as defined in section 2. This will need suitable modifications in the proof of Theorem 1 and 2, and we leave the details to the reader.

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