ON THE UNIFICATION OF BERNOULLI AND EULER POLYNOMIALS

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In this paper the Bernoulli and Euler polynomials, the study of which has done much good in the field of theory of numbers, have been unified. The generating function technique has been used to obtain pure and mixed recurrence relations and many more generalizations investigated by Nörlund (1924) and Carlitz (1962) in the unified form. Incidently the present approach also leads to the generalization of Carlitz's 'Eulerian polynomials'. Our polynomials also give as special cases the Bernoulli and Genocchi numbers.

1. INTRODUCTION

The Bernoulli polynomials $\{B_n(x)\}$, n=0, 1, 2, ... are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\ \frac{t^n}{n!}$$

and the Bernoulli numbers are given by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Similarly the Euler polynomials $\{E_n(x)\}$, $n=0, 1, 2, \ldots$; are defined by

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty} E_n(x)\ \frac{t^n}{n!}.$$

Carlitz (1962) defined Eulerian polynomials $\{\Phi_n(x, \xi)\}$ by

$$\frac{1-\xi}{1-\xi e^t}\, e^{xt}=\sum_{n=0}^{\infty} \Phi_n(x, \xi)\ \frac{t^n}{n!}$$

with $\xi \neq 1$, but otherwise arbitrary. They are connected with Euler polynomials by the relation

$$\Phi_n(x, -1) = E_n(x) .$$

The purpose of this paper is to unify the polynomials of Bernoulli, Euler and Eulerian polynomials. Our approach also gives as special cases the Bernoulli numbers and Genocchi numbers for $x=0$.

We define the polynomials $D_n(x; a, k)$, where a is a non-zero real number and k is an integer, by the following generating relation

$$\frac{2\left(\frac{t}{2}\right)^k e^{xt}}{e^t - a} = \sum_{n=0}^{\infty} D_n(x; a, k) \frac{t^n}{n!} . \qquad \ldots(1.1)$$

We state the following relationship between our polynomials $D_n(x; a, k)$ and Bernoulli and other polynomials.

(i) *Bernoulli polynomials* : When $a = k = 1$, we have

$$D_n(x; a, k) = B_n(x). \qquad \ldots(1.2)$$

(ii) *Bernoulli numbers* : When $a = k = 1$, and $x = 0$ we have

$$D_n(0; 1, 1) = B_n .$$

(iii) *Euler polynomials* : When $a = -1, k = 0$, we have

$$D_n(x; a, k) = E_n(x) . \qquad \ldots(1.3)$$

(iv) *Genocchi numbers* : When $-a = k = 1, x = 0$, we get

$$2 D_n(0; -1, 1) = G_n$$

where these numbers are defined by

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} .$$

(v) *Eulerian polynomials* : When we put $k = 0$, $a = 1/\xi$, with $\xi \neq 1$, we get

$$D_n\left(x; \frac{1}{\xi}, 0\right) = \frac{2\xi}{\xi-1} \Phi_n(x, \xi) .$$

2. Some Results in the Unified Form

Differentiating both the sides of (1.1) with respect to x, we get

$$\sum_{n=0}^{\infty} D_n(x; a, k) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} D_n'(x; a, k) \frac{t^n}{n!}$$

where prime denotes the differentiation with respect to x. Equating the coefficients of t^n on both the sides we have the following recurence relation

$$D_n'(x; a, k) = n\, D_{n-1}(x; a, k). \qquad \ldots(2.1)$$

It is also evident from (1.1) that

$$\sum_{n=0}^{\infty} \left\{ D_n(x+1; a, k) - aD_n(x; a, k) \right\} \frac{t^n}{n!}$$

$$= 2\left(\frac{t}{2}\right)^k e^{xt}$$

$$= \frac{1}{2^{k-1}} \sum_{n=k}^{\infty} \frac{x^{n-k}}{(n-k)!}\, t^n \,.$$

Equating the coefficients of t^n on both the sides we are led to

$$D_n(x+1; a, k) - aD_n(x; a, k)$$

$$= \frac{1}{2^{k-1}} \binom{n}{k} k!\, x^{n-k}, \text{ with } n \geqslant k\,. \qquad \ldots(2.2)$$

The result (1.1) can also be transcribed as

$$\sum_{n=0}^{\infty} D_n(x+1; a, k) \frac{t^n}{n!} = \frac{2\,(t/2)^k\, e^{xt+t}}{e^t - a}$$

$$= \frac{2\,(t/2)^k\, e^{xt}}{e^t - a} \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$= \sum_{r=0}^{\infty} D_r(x; a, k) \frac{t^r}{r!} \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} D_r(x; a, k)\, \frac{t^n}{n!}\,.$$

This relation explicitly implies that

$$D_n(x+1; a, k) = \sum_{r=0}^{n} \binom{n}{r} D_r(x; a, k) . \qquad \ldots(2.3)$$

Elimination of $D_n(x+1; a, k)$ from (2.2) and (2.3) leads to

$$\sum_{r=0}^{n} \binom{n}{r} D_n(x; a, k) \frac{1}{2^{k-1}} \binom{n}{k} k!\, x^{n-k} + aD_n(x; a, k) . \qquad \ldots(2.4)$$

From (1.1) it is also possible to write

$$\sum_{n=0}^{\infty} D_n(x; \frac{1}{a}, k) \frac{t^n}{n!} = \frac{2a\left(\frac{t}{2}\right)^k e^{xt}}{ae^t-1} \qquad \ldots(2.5)$$

and also

$$\sum_{n=0}^{\infty} D_n(1-x; a, k) \frac{t^n}{n!} = -\frac{2\,(t/2)^k\, e^{t-xt}}{e^t-a} . \qquad \ldots(2.6)$$

Using (2.5) and (2 6) and as well changing the sign of t we get the symmetric relation

$$D_n(1-x; a, k) = \frac{1}{a} (-1)^{k+1+n} D_n(x; \frac{1}{a}, k) . \qquad \ldots(2.7)$$

Replacing x by $-x$ in (2.2) and eliminating $D_n(1-x; a, k)$ from (2.7) we get

$$(-1)^{n+k+1} a^2 D_n(-x; a, k)$$
$$= D_n(x; \frac{1}{a}, k) + \frac{a}{2^{k-1}} \binom{n}{k} k!\, x^{n-k} . \qquad (2.8)$$

The result (1.1) allows us to write

$$\sum_{n=0}^{\infty} D_n(x; a^2, k) \frac{t^n}{n!} = \frac{2\left(\frac{t}{2}\right)^k e^{xt}}{e^t-a^2} .$$

Replacing t by $2t$ we get

$$\sum_{n=0}^{\infty} D_n(x; a^2, k) \frac{2^n t^n}{n!} = \frac{2^{k-1}}{e^t-a} \frac{2\left(\frac{t}{2}\right)^{k-1} e^{2tx}}{e^t+a} .$$

On account of (1.1) the right-hand side of this expression can also be written as

$$\sum_{n=0}^{\infty} D_n(x;\, a^2,\, k)\, \frac{2^n\, t^n}{n!}$$

$$= 2^{k-1} \sum_{n=0}^{\infty} D_n(0;\, a,\, 1)\, \frac{t^n}{n!} \sum_{r=0}^{\infty} D_r(2x;\, -a,\, k-1)\, \frac{t^r}{r!}$$

$$= 2^{k-1} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left\{ \frac{D_{n-r}(0;\, a,\, 1)\; D_r(2x;\, -a,\, k-1)}{r!\,(n-r)!} \right\} t^n .$$

This gives the following interesting result

$$D_n(x;\, a^2,\, k) = \frac{2^{k-1}}{2^n} \sum_{r=0}^{n} \binom{n}{r} D_{n-r}(0;\, a,\, 1)\; D_r(2x;\, -a,\, k-1). \qquad \ldots (2.9)$$

Let us now consider the identity

$$\frac{2(t/2)^k \exp\left(\frac{xt}{2}\right)}{\exp(t/2)+a} = \frac{2\left(\frac{t}{2}\right)^k \exp\left[\frac{t(x+1)}{2}\right]}{\exp(t)-a^2} - \frac{2a(t/2)^k \exp\left(\frac{xt}{2}\right)}{\exp(t)-a^2} .$$

Because of the generating relation (1.1) we now get

$$\sum_{n=0}^{\infty} D_n(x;\, -a,\, k-1) \frac{2^{k-1}}{2^{n+1}} \frac{t^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} D_n\left(\frac{x+1}{2};\, a^2,\, k\right) \frac{t^n}{n!} - a \sum_{n=0}^{\infty} D_n\left(\frac{x}{2};\, a^2,\, k\right) \frac{t^n}{n!} .$$

Equating the coefficients of r^n on both the sides we get

$$D_{n-1}(x;\, -a,\, k-1)\, 2^{k-1}$$

$$= \frac{2^n}{n} \left\{ D_n\left(\frac{x+1}{2};\, a^2,\, k\right) - aD_n\left(\frac{x}{2};\, a^2,\, k\right) \right\}. \qquad \ldots (2.10)$$

3. Particular Cases

Here we enlist the results for the Bernoulli and Euler polynomials as particular cases of our results obtained in §2.

CASE I : *Bernoulli polynomials*—On account of the relation (1.2) we get from the results (2.1), (2.2), (2.4), (2.7) and (2.8) after effecting the substitutions $a = k = 1$ in each, the following results for the Bernoulli polynomials respectively.

$$B'_n(x) = n\,B_{n-1}(x),$$

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

$$\sum_{r=0}^{n-1} \binom{n}{r} B_r(x) = nx^{n-1},$$

$$B_n(1-x) = (-1)^n\,B_n(x),$$

and

$$(-1)^n B_n(-x) = nx^{n-1} + B_n(x).$$

CASE II : *Euler polynomials*—In view of the relationship (1.3) between our polynomials and Euler polynomials we get from (2.1), (2.2), (2.4), (2.7) and (2.8) after making the substitutions $a = -1$, $k = 0$ in each the following results involving the Euler polynomials respectively.

$$E'_n(x) = nE_{n-1}(x),$$

$$E_n(x+1) + E_n(x) = 2x^n,$$

$$\sum_{r=0}^{n} \binom{n}{r} E_r(x) + E_n(x) = 2x^n,$$

$$E_n(1-x) = (-1)^n\,E_n(x),$$

and

$$(-1)^{n+1}\,E_n(-x) = E_n(x) - 2x^n.$$

It is of interest to observe that, if we put $a=k=1$ in (2.9), we get the usual relation between the Bernoulli polynomials and Euler polynomials in the form

$$B_n(x) = \frac{1}{2^n} \sum_{r=0}^{n} \binom{n}{r} B_{n-r}\,E_r(2x)$$

where $\{B_{n-r}\}$ are Bernoulli numbers. In a similar manner this very substitution in (2.10) also yields the well-known relationship between the Euler polynomials and Bernoulli polynomials as

$$E_{n-1}(x) = \frac{2^n}{n}\left\{ B_n\left(\frac{x+1}{2}\right) - B_n(x) \right\}.$$

4. Nörlund's Results

Nörlund (1924, chapt. II) states the following results for Bernoulli and Euler polynomials :

$$\sum_{s=0}^{n-1} B_r\left(x+\frac{s}{n}\right)=n^{1-r}B_r(nx), \qquad \text{... (4.1)}$$

$$\sum_{s=0}^{n-1} (-1)^s E_r\left(x+\frac{s}{n}\right)=n^{-r}\ E_r(nx)\ ,\ (n\ \text{odd}), \qquad \text{... (4.2)}$$

$$\sum_{s=0}^{n-1} (-1)^s\ B_{r+1}\left(x+\frac{s}{n}\right)=-\frac{(r+1)}{2n^r}\ E_r(nx),\ (n\ \text{even}). \qquad \text{... (4.3)}$$

We establish these results by our approach. Starting with (1.1) we get

$$\sum_{r=0}^{\infty}\frac{(nt)^r}{r!}\sum_{s=0}^{n-1} D_r\left(x+\frac{s}{n};\ a^n,\ k\right)$$

$$=\sum_{s=0}^{n-1}\frac{2n^k\left(\frac{t}{2}\right)^k \exp\ (nxt+st)}{(\exp\ (nt)-a^n).\ a^s}$$

$$=\frac{n^k\ 2\left(\frac{t}{2}\right)^k \exp\ (nxt)}{\exp\ (t)-a}\cdot\frac{\exp\ (t)-a}{\exp\ (nt)-a^n}\sum_{s=0}^{n-1}\frac{\exp\ (st)}{a^s}$$

$$=\frac{n^k}{a^{n-1}}\sum_{r=0}^{\infty} D_r\ (nx;\ a,\ k)\frac{t^r}{r!}\ \cdot$$

Equating the coefficients of t^r on both the sides we get

$$\sum_{s=0}^{n-1} D_r\left(x+\frac{s}{n};\ a^n,\ k\right)=\frac{n^{k-r}}{a^{n-1}}D_r\ (nx;\ a,\ k) \qquad \text{... (4.4)}$$

Putting $a=k=1$ (4.4) reduces to (4.1) and putting $a=-1$, $k=0$ and for n being odd (4.4) reduces to (4.2).

Similarly it follows from (1.1) that

$$\sum_{r=0}^{\infty}\frac{(nt)^r}{r!}\sum_{s=0}^{n-1}\frac{1}{a^s}\,D_r\left(x+\frac{s}{n}\,;\,a^n,\,k\right)$$

$$=\sum_{s=0}^{n-1}\frac{2n^k}{\exp\,(nt)-a^n}\cdot\frac{\left(\frac{t}{2}\right)^k\exp\,(nxt)\exp\,(st)}{a^s}$$

$$=\frac{2n^k\left(\frac{t}{2}\right)^k\exp\,(nxt)}{a^{n-1}(e^t-a)}$$

$$=\frac{n^k}{2a^{n-1}}\cdot\frac{2t\left(\frac{t}{2}\right)^{k-1}\exp\,(nxt)}{e^t-a}\,.$$

The use of (1.1) again gives us

$$\sum_{r=0}^{\infty}\frac{(nt)^r}{r!}\sum_{s=0}^{n-1}\frac{1}{a^s}\,D_r\left(x+\frac{s}{n}\,:\,a^n,\,k\right)$$

$$=\frac{n^k}{2a^{n-1}}\sum_{r=0}^{\infty}D_r\,(nx;\,a,\,k-1)\,\frac{t^{r+1}}{r!}\,.$$

Hence we have

$$\sum_{s=0}^{n-1}\frac{1}{a^s}\,D_{r+1}\left(x+\frac{s}{n}\,;\,a^n,\,k\right)=\frac{(r+1)}{2n^{r+1}}\frac{n^k}{a^{n-1}}\,D_r\,(nx;\,a,\,k-1).$$

The substitutions $-a=k=1$ transform this to the result (4.3) when n is even.

5. Carlitz's Generalizations

Carlitz (1962) gave generalizations of the multiplication formulae (4.1) to (4.3) as

$$n^{r-1}\sum_{s=0}^{n-1}B_r\left(\frac{x}{n}+\frac{ms}{n}\right)=m^{r-1}\sum_{p=0}^{m-1}B_r\left(\frac{x}{m}+\frac{np}{m}\right) \qquad \text{...(5.1)}$$

$$n^r\sum_{s=0}^{n-1}(-1)^sE_r\left(\frac{x}{n}+\frac{ms}{n}\right)=m^r\sum_{p=0}^{m-1}(-1)^pE_r\left(\frac{x}{m}+\frac{np}{m}\right),$$

$(m,\ n$ both odd) ...(5.2)

$$n^r\sum_{s=0}^{n-1}(-1)^s B_{r+1}\left(\frac{x}{n}+\frac{ms}{n}\right)$$

$$=\frac{-1}{2}(r+1)m^r\sum_{p=0}^{m-1}E_r\left(\frac{x}{m}+\frac{np}{m}\right),\ (n\text{ even}) \qquad \ldots(5.3)$$

respectively

We prove the results as follows. On account of (1.1) we have

$$\sum_{r=0}^{\infty}\frac{(nt)^r}{r!}\sum_{s=0}^{n-1}\frac{1}{a^{sm}}D_r\left(\frac{x}{n}+\frac{sm}{n};\ a^n, k\right)$$

$$=\sum_{s=0}^{n-1}\frac{1}{a^{sm}}\frac{2\left(\frac{nt}{2}\right)^k e^{xt}e^{mst}}{e^{nt}-a^n},$$

$$=\frac{n^k\,2\left(\frac{t}{2}\right)^k e^{xt}\left(e^{mnt}-a^{mn}\right)}{a^{m(n-1)}\left(e^{nt}-a^n\right)\left(e^{mt}-a^m\right)}. \qquad \ldots(5.4)$$

From (5.4) and using the symmetry in m and n, we obtain

$$a^{m(n-1)}n^{r-k}\sum_{s=0}^{n-1}\frac{1}{a^{sm}}D_r\left(\frac{x}{n}+\frac{sm}{n};\ a^n, k\right)$$

$$=a^{n(m-1)}m^{r-k}\sum_{p=0}^{m-1}\frac{1}{a^{np}}D^r\left(\frac{x}{m}+\frac{np}{m};\ a^m,\ k\right). \qquad \ldots(5.5)$$

Putting $a=k=1$ we get the result (5.1). For m, and n both odd, $a=-1$, $k=0$ the result (5.5) is reduced to (5.2).

In order to obtain (5.3) we have because of (1.1) the relation

$$\sum_{r=0}^{\infty}\frac{(mt)^r}{r!}\sum_{p=0}^{m-1}\frac{1}{a^{np}}D_r\left(\frac{x}{m}+\frac{np}{m};\ a^m,\ k-1\right)$$

$$=\frac{2(t/2)^{k-1}\exp(xt)}{\exp(mt)-a^m}\sum_{p=0}^{m-1}\frac{\exp(npt)}{a^{np}}$$

$$=\frac{2\left(\frac{t}{2}\right)^{k-1}\exp(xt)\left(\exp(mnt)-a^{mn}\right)}{a^{n(m-1)}\left(\exp(mt)-a^m\right)\left(\exp(nt)-a^n\right)}. \qquad \ldots(5.6)$$

From (5.4) and (5.6) we can now obtain the following result :

$$\frac{n^k m^r}{2^{k-1}\, a^{m(n-1)}\, r!} \sum_{p=0}^{m-1} \frac{1}{a^{np}}\, D_r\left(\frac{x}{m} + \frac{np}{m};\, a^m,\, k-1 \right)$$

$$= \frac{m^{k-1}}{2^{k-2}}\ \frac{n^{r+1}}{a^{n(m-1)}\,(r+1)!} \sum_{s=0}^{n-1} D_{r+1}\left(\frac{x}{n} + \frac{ms}{n};\, a^n,\, k \right)$$

and also

$$\frac{n^{r+1-k}}{a^{n(m-1)}} \sum_{s=0}^{n-1} \frac{1}{a^{ms}} D_{r+1}\left(\frac{x}{n} + \frac{ms}{n};\, a^n,\, k \right) \qquad \text{.. (5.7)}$$

$$= \frac{1}{2}\ \frac{(r+1)}{a^{m(n-1)}}\, m^{r+1-k} \sum_{p=0}^{m-1} \frac{1}{a^{np}} D_r\left(\frac{x}{m} + \frac{np}{m};\, a^m,\, k-1 \right). \qquad \text{...(5.7)}$$

When $-a=k=1$ and for even n and odd m (5.7) yields (5.3) as a particular case.

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