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A FAMILY OF HYPO-HAMILTONIAN GENERALIZED PRISMS

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In this paper we construct a family of hypo-Hamiltonian generalized prisms with $4k + 2$ vertices $k \neq 1, 3$. This family gives us new cubic hypo-Hamiltonian graphs for $k > 5$.

INTRODUCTION

In this paper we consider undirected graphs and call the number of vertices of a graph G as the order of G. A graph G is Hamiltonian if it has a circuit passing through every vertex of G. It is hypo-Hamiltonian if it is not Hamiltonian but every vertex deleted subgraph $G - v$ is Hamiltonian. As minimum degree in a hypo-Hamiltonian graph is three cubic hypo-Hamiltonian graphs are minimally hypo-Hamiltonian. These graphs have been obtained by Bondy (1972) and Chvatal (1973). Here we consider the existence of cubic. Hypo-Hamiltonian graphs in the special context of permutation graphs.

The concept of permutation graph of a graph G was introduced by Chartrand and Harary (1967). Let G be a graph with $V(G) = \{1, 2, ..., n\}$ and let $\pi = (\pi(1), \pi(2) ... \pi(n))$ be a permutation on $V(G)$. Let G_1 and G_2 be two vertex disjoint copies of G such that the vertex i of G is labelled a_i in G_1 and b_i in G_2. Then the permutation graph (G, π) of the graph G, also written as $P_\pi(G)$ or $G(\pi)$, consists of the two graphs G_1, G_2 and n additional edges $\{a_i, b_{\pi(i)}\}_{i=1}^{n}$ joining them.

Klee (1972) constructed permutations π for which (c_n, π) $(n \geqslant 3)$ is non-Hamiltonian. They are called generalized n-prisms there. He proved the existence of a non-Hamiltonian generalized prism $4k + 2$ vertices for all $k \geqslant 2$, $k \neq 3$. Here we prove that, in fact, we have hypo-Hamiltonian generalized prisms in each case. In particular we get new cubic hypo-Hamiltonian graphs of order $4k + 2, k > 5$. This family gives us two new cubic hypo-Hamiltonian graphs of order 30, i.e. $G(F_8, F_8, F_8)$ and $(c_{15}, \bar{\pi}_{2,11}, \bar{\pi}_{2,5}(1))$.

PRELIMINARIES

The permutation $\pi_{m,n}$ is given by

$$\begin{cases} \pi_{m,n}(i) = \text{residue of } im \pmod{n} \text{ if } 1 \leqslant i \leqslant n-1 \\ \pi_{m,n}(n) = n, \end{cases}$$

where m and n are relatively prime and $1 \leqslant m < n/2$.

The generalized Petersen graph $G(n, d)$ has $V(G(n, d)) = \{u_i, v_i\}_{i=1}^{n}$ and $E(G(n, d)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+d}\}_{i=1}^{n}$ where $1 \leqslant d \leqslant n-1$, $n \neq 2d$ and all the suffices are to be read modulo n. We have the following:

Theorem 1 — $G(n, m) \simeq (c_n, \pi_{m,n})$ if $(m, n) = 1$.

PROOF : Let $a_1 a_2 \ldots a_n a_1$ and $b_1 b_2 \ldots b_n b_1$ be consecutive labellings along the two cycles c_n. Let k be such that $km \equiv 1 \pmod{n}$, The map $u_i \to b_i$ and $v_i \to a_{ki}$ is seen to be an isomorphism.

Robertson (1968) and Bondy (1972) have established that $G(n, 2)$ is non-Hamiltonian if and only if $n \equiv 5 \pmod 6$. Hence, $(c_n, \pi_{2,n})$ is non-Hamiltonian if and only if $n \equiv 5 \pmod 6$.

Let $\pi_1 \in S_{n_1}$ and $\pi_2 \in S_{n_2}$. Then catenate $\pi = (\pi_1, \pi_2) \in S_{n_1+n_2}$ where $\pi(i) = \pi_1(i)$ for $1 \leqslant i \leqslant n_1$ and $\pi(i) = n_1 + \pi_2(i-n)$ for $n_1 < i \leqslant n_1 + n_2$. The operation of catenation can be extended in an obvious way to an arbitrary finite sequence of permutations. If $\pi \in S_{n-1}$ then $(\pi, (1)) \in S_n$ is denoted by π'. If $\pi \in S_n$ and $\pi(n) = n$, then $\bar{\pi}$ denotes the restriction of π to $\{1, 2, 3, \ldots, n-1\}$.

Let G be a graph. A pair (a, b) of vertices of G is called 'good' in G if G has a spanning path with end points a and b. Similarly, $((a, b), (c, d))$ is called good in G if G has a spanning subgraph consisting of two vertex disjoints paths one having end points a and b and the other c and d. Klee (1972) called a permutation $\pi \in S_n$ 'bad' if none of (a_1, b_1), (a_n, b_n), (a_1, b_n), (a_n, b_1) $((a_1, b_n), (a_n, b_1))$ and $((a_1, b_1), (a_n, b_n)$ is 'good' in the graph (P_n, π). Otherwise π is 'good'. He proved the following:

Theorem 2 (Klee 1972) — If $\pi \in S_n (n \geqslant 3)$, $\pi(n) = n$ then (c_n, π) is non-Hamiltonian if and only if $\bar{\pi}$ is bad.

Theorem 3 (Klee 1972) — If $\pi_i \in S_{n_i}$ for $1 \leqslant i \leqslant k$ and the following three conditions hold then the catenate

$$\pi = (\pi_1, \pi_2, \ldots, \pi_k) \text{ is a bad permutation.}$$

(a) for each i, either $n_i = 1$ or π_i is bad

(b) there is an even number (possibly zero) of 1's among the n_i's.

(c) $1 < n_1$, $1 < n_k$ and no two 1's among the n_i's appear consecutively.

Theorem 4 (Klee 1972) — For odd $n \geqslant 3$ there exists a generalized n-prism not admitting a Hamiltonian cycle if and only if n is neither 3 nor 7.

The graphs for $n \neq 11$ in Theorem 4 were obtained by taking $\pi_i = \bar{\pi}_{2,5} = (2413)$ for $n_i > 1$ in Theorem 3 suitably. For $n = 11$, the example of $(c_{11}, \pi_{2,11})$ was given.

The first graph $(c_5, (24135))$ in the sequence of graphs in Theorem 4 is hypo-Hamiltonian as it is the well-known Petersen graph. The second graph

$$(c_9, \bar{\pi}_{2,5}, \bar{\pi}_{2,5}, (1))$$

is also hypo-Hamiltonian as it is isomorphic to Sousselier's hypo-Hamiltonian graph on eighteen vertices. This observation helped us to construct a new family.

We call $\pi \in S_n$ hypo-good if

(i) $\bar{\pi}$ is bad, (ii) both (a_1, a_n) and (b_1, b_n) are good in (P_n, π), (iii) for each vertex v of (P_n, π) at least one of (a_1, b_n) (a_n, b_1), $((a_1, b_1), (a_n, b_n))$, $((a_1, b_n), (a_n, b_1))$ is good in $(P_n, \pi) - v$.

Theorem 5 — $\pi_{2,n}$ is hypo-good for all $n \equiv 5 \pmod 6$.

PROOF: Let $n = 6m + 5$, $m \geqslant 0$. By Theorem 1, $(c_n, \pi_{2,n})$ is non-Hamiltonian and hence $\bar{\pi}_{2,n}$ is a bad permutation by Theorem 2. Thus the first condition holds. Now we show that conditions (ii) and (iii) also hold.

Let $G = (P_{6m+4}, \bar{\pi}_{2,6m+5})$. The pairs (a_1, a_{6m+4}) and (b_1, b_{6m+4}) are good in G for

$$a_1 . a_2 \ldots a_{3m+2} b_{6m+4} b_{6m+3} \ldots b_1 a_{3m+3} a_{3m+4} \ldots a_{6m+4} \text{ and}$$

$$b_1 a_{3m+3} a_{3m+4} \ldots a_{6m+4} b_{6m+3} b_{6m+2} \ldots b_2 a_1 a_2 \ldots a_{3m+2} b_{6m+4}$$

are spanning paths in G. This proves (11).

Let v be any vertex of G. Now we show that one of (a_1, b_{6m+4}), (a_{6m+4}, b_1), $((a_1, b_1), (a_{6m+4}, b_{6m+4}))$ $((a_1, b_{6m+4}), (a_{6m+4}, b_1))$ is good in $G - v$. We introduce the following notation for the description of a path P in G. A sequence $(d_1, d_2, \ldots \overleftarrow{d_i}, d_{i+1}, \ldots d_t)$ of positive integers d_i denotes a path P in G which follows the a-path P_1 and b-path P_2 in G alternately, the number of vertices in the successive intercepts being as prescribed by the sequence. A d_i with an arrow above means P passes through d_i consecutive vertices of P_i, $i = 1, 2$, in a leftward direction while a d_i without an arrow above means that the path P follows the P_i, $i = 1, 2$ in a rightward direction for d_i vertices.

First we consider deletion of vertices from the a-path $G - a_i$: (b_1, a_{6m+4})-path is given by $(4, 3, \ldots, 3, 4, 3, 3, \ldots, 3, 2)$ where 3 occurs $2m$ times after the first 4 and $(2m - 1)$ times after the second 4.

$G - a_2 : (a_1, b_{6m+4})$-path is given by $(1, 2, 2, 3, 3, \overleftarrow{4}, 3, 2, 3, \overleftarrow{4}, 3, 2, 3, \overleftarrow{4}, 3, \ldots, 2, 3, \overleftarrow{4}, 3, 1)$ where 2, 3, 4, 3 group occurs $(m - 1)$ times.

$G - a_k : 3 \leqslant k \leqslant 3m + 1$.

(a_1, b_1)-path is given by $(k - 1, \overleftarrow{2k} - 2)$ and (a_{6m+4}, b_{6m+4})-path is given by $(\overleftarrow{3m - k} + 2, \overleftarrow{3}, \overleftarrow{3m + 2}, 6m - 2k + 3)$.

$G - a_{3m+2} : (a_1; b_{6m+4})$-path is given by $(3m + 1, \overleftarrow{6m + 2}, 3m + 2, 2)$. The paths for remaining a_i, $3m + 3 \leqslant i \leqslant 6m + 4$ can be written out by symmetry.

Next we consider deletion of vertices from the b-path. $G - b_1 : (a_1, b_{6m+4})$-path is given by $(1, 3, 3, \ldots, 3, 2, \overleftarrow{2}, \overleftarrow{3}, \overleftarrow{3}, \ldots, \overleftarrow{3}, \overleftarrow{4}, 1)$ where 3 occurs $2m$ times and $\overleftarrow{3}$ occurs $2m - 1$ times.

The path for $G - b_{6m+4}$ can be written by symmetry.

$G - b_{2k}$, $1 \leqslant k \leqslant 3m + 1 : (a_1, b_1)$-path is given by $(3m + k + 2, \overleftarrow{2k - 1})$ and (a_{6m+4}, b_{6m+4})-path is given by $(\overleftarrow{3m - k} + 2, 6m - 2k + 4)$

$G - b_{2k+1}$, $1 \leqslant k \leqslant 3m + 1 : (a_1, b_1)$-path is given by $(k, \overleftarrow{2k})$ and (a_{6m+4}, b_{6m+4})-path is given by $(\overleftarrow{6m - k} + 4, 6m - 2k + 3)$.

This completes the proof.

Theorem 6 — Let π_i be an n_i-permutation for $1 \leqslant i \leqslant k$ and the following three conditions hold

(a) for each i, either $n_i = 1$ or π_i is hypo-good number,

(b) there is an even (possibly zero) number of 1's among the n_i's,

(c) $1 < n_1$, $1 < n_k$ and no two 1's among the n_i's appear consecutively.

Then the generalized prism (c_n, π') is hypo-Hamiltonian where $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ and $n = \sum_{i=1}^{k} n_i + 1$.

PROOF: Let $G = (c_n, \pi')$. Since each π_i with $n_i > 1$ is hypo-good, it is bad and hence $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is a bad permutation by Theorem 3. Therefore G is non-Hamiltonian by Theorem 2. We show below that every vertex-deleted subgraph $G - v$ of G is Hamiltonian. We call the induced subgraph (P_{n_i}, π_i) of G a 'block' if $n_i > 1$ and a 'single edge' if $n_i = 1$.

Let v be any vertex of G. It is either in a block or in a single edge. We consider each case separately.

Case I — Let $V = a_i(b_i)$ where a_ib_i is a singe edge. Then starting from $b_i(a_i)$ follow the paths (ii) in the hypo-good definition through each block and the single edges otherwise. This traces a Hamiltonian cycle of $G - v$ since the number of single edges in $G - v$ is even.

Case II — Let v be a vertex in a block $H = (P_{n_i}, \pi_i)$. Let the end points of H be $\{a_l, b_l, a_m, b_m\}$, Let (a_l, b_m) or (a_m, b_l) be good in $H - v$. These paths can be extended to a Hamiltonian cycle of $G - v$ by using the paths (ii) in definition for the remaining blocks and single edges otherwise as $G - v$ has an odd number of single edges.

If $((a_l, b_l), (a_m, b_m))$ or $((a_l, b_m), (a_m, b_l))$ is good in $H - v$, then these paths can be extended to a Hamiltonian cycle of $G - v$ by using the rest of the a-cycle for going from a_l to a_m and rest of the b-cycle for going from b_l to b_m. This completes the proof of the theorem.

Theorem 7 — For odd $n \geqslant 3$, there exists a hypo-Hamiltonian generalized n-prism if and only if n is neither 3 nor 7.

PROOF : The proof follows from Theorems 4, 5 and 6.

Let the family of cubic hypo-Hamiltonian graphs constructed by Bondy (1972), namely $G(6m + 5, 2)$, $m \geqslant 0$ be denoted by A, the family constructed by Chvatal (1973) by B and the one constructed here in Theorems 5 and 6 by C. Clearly $A \subset C$. Now we examine how the families B and C are related.

The family B was constructed by using the special graphs called flip-flops defined as follows:

A flip-flop is a quintuple (G, a, b, c, d) where G is a graph and a, b, c, d are distinct vertices of G such that (I) (a, d), (b, c) and $((a, d), (b, c))$ are good in G.

(II) none of (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$ and $((a, c), (b, d))$ is good in G.

(III) for every vertex u of G at least one of (a, c), (b, d), $((a, b), (c, d))$, $((a, c), (b, d))$ is good in $G - u$.

The order of flip-flop is the number of vertices in G. A flip-flop is called cubic if a, b, c, d have degree two in G and all other points have degree three.

Let $F_i = (G_i, a_i, b_i, c_i, d_i)$ be flip-flops. (F_1, F_2) denotes the quintuple $(G_4, a_1, b_1, c_2, d_2)$ where G_4 is obtained by taking (disjoints) graphs G_1, G_2 and joining c_1 to b_2 and d_1 to a_2. (F_1, F_2, F_3) denotes the quintuple $(G_5, a_1, b_1, c_3, d_3)$ where G_5 is obtained as follows: take (pairwise disjoint) graphs G_1, G_2, G_3, add four more vertices u_1, v_1, u_2, v_2 and ten more lines u_1v_1, u_2v_2, c_1u_1, u_1a_2, d_1v_1, v_1b_2, c_2u_2, u_2a_3, d_2v_2, v_2b_3. It was proved that (F_1, F_2) and (F_1, F_2, F_3) are flip-flops.

Let H be a graph and a, b, c, d be four distinct vertices of H. The graph $G(H)$ based on H is obtained from H by adding two more vertices u, v and five more lines uv, ua, ud, vb and vc. It was shown that if F is a flip-flop then $G(F)$ is hypo-Hamiltonian. The family B was constructed by using two flip-flops F_8 and F_{26} where $G(F_8)$ is the petersen graph and $G(F_{26})$ is the Coexter's hypo-Hamiltonian graph on 28 vertices.

Theorem 8 — $G(F_{26})$ is not a generalized prism.

PROOF: $G = G(F_{26})$ based on F_{26} is the coexter's graph on 28 vertices. We take its description given in (Klee) namely,

$$V(G) = \{a_i, b_i, c_i, d_i\}_{i=1}^{7}, \; E(G) = \{a_i a_{i+1}, b_i b_{i+2}, c_i c_{i+3},$$

$$a_i d_i, b_i d_i, c_i d_i\}_{i=1}^{7}.$$

If possible let G be a generalized prism. So G consists of two vertex disjoint 14 cycles C_1 and C_2 such that each vertex of C_1 is adjacent to exactly one vertex of C_2. Since there are seven vertices d_i in G, one of these two cycles say c_1 must contain four of these vertices d_i. Let these vertices be di_1, di_2, di_3 and di_4. Then c_1 is of the form

$$\ldots x_{i_1} d_{i_1} y_{i_1} \ldots x_{i_2} d_{i_2} y_{i_2} \ldots x_{i_3} d_{i_3} y_{i_3} \ldots x_{i_4}, d_{i_4} y_{i_4} \ldots$$

where $x, y \in \{a, b, c\}$. Since i_1, i_2, i_3, i_4 are distinct the 14-cycle C_1 can use at most two more suffices $i = 1, 2, \ldots, 7$. Therefore, there exists a suffix $j \in \{1, 2, \ldots 7\}$ such that d_j and all its three adjacent vertices a_j, b_j, c_j lie in C_2. This is a contradiction. Hence $G(F_{26})$ is not a generalized prism.

Theorem 9 — Let F_i be cubic flip-flops of order n_i for $1 \leqslant i \leqslant k$. Let $F = (F_1, F_2, \ldots, F_k)$. If $G(F)$ is a generalized prism, then so is each $G(F_i)$ where $i \in \{1, 2, \ldots, k\}$.

PROOF: Let $G(F)$ be a generalized prism and let C_1 and C_2 be the a-cycle and b-cycle of G respectively. Let $F_j = (G_j, a_j, b_j, c_j, d_j)$, $1 \leqslant j \leqslant k$ be any flip-flop in F considered as an induced subgraph of $G(F)$ in the natural way. Then a_j, b_j, c_j, d_j are the only vertices of F_j having an adjacency in $G(F) - G_j$. Since these four vertices have only one adjacency each in $G(F) - G_j$ the intersections of C_1 and C_2 with G_j are connected graphs and hence form two vertex disjoint n_i-paths in G_j. Rest of the proof follows from the fact that $G(F)$ is a generalized prism.

Theorem 10 — The graphs of B formed from F_8 alone are in C.

PROOF: It is easy to exhibit isomorphism to see that $G(F_8)$, $G(F_8, F_8)$ and $G(F_8, F_8, F_8)$ are generalized prisms.

From the above three theorems we see that only graphs in B which are generalized prisms are those formed from F_8 alone and all of these are in C. Thus the family C contains all generalized prisms present among the two earlier families A and B. As the family C contains many more graphs besides these all of them are new cubic hypo-Hamiltonian graphs of order $4k + 2$.

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