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OSCILLATION OF A SPHERE ALONG THE AXIS OF A ROTATING FLUID

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The flow created in an inviscid, incompressible, unbounded rotating fluid by the harmonic oscillations of a sphere along the axis of rotation is considered. The analysis indicates two types of flows—elliptic or hyperbolic according as $\sigma >$ or $< 2\Omega$, where σ is the forcing frequency and Ω the angular velocity of rotation. In the former there is potential flow and in the latter the flow consists of superposition of two wave-trains. In elliptic case when $\sigma \to 2\Omega$, the system resonates and the velocities alogg r and θ directions become indefinitely large. In hyperbolic case when $\sigma \to 2\Omega$, the wave speed drops to zero and the crests lie in the plane $z = 0$.

When $\sigma = 2\Omega$ (parabolic case) no definite conclusions can be drawn from the linearized equations of motion.

1. Introduction

The oscillation of a sphere in a rotating fluid was first considered by Gortler (1944) and also studied by Morgan (1951) and Oser (1957, 1958). All these contributions are based on linearized equations governing axisymmetric perturbations to a basic rotation with specialization (introduced by Gortler) of harmonically time-dependent perturbations. Reynolds (1962a, b) considered an unsteady axisymmetric flow generated in an inviscid, incompressible rotating liquid by small harmonic oscillations of a body immersed in the liquid. The analysis suggests the existence of two distinct regimes, one corresponding to high frequency oscillation or small rotation and the other corresponding to slow forcing oscillation or rapid rotation. These are respectively the elliptic and hyperbolic regimes of the equations obtained by Gortler. Unlike Gortler's prediction Reynolds reveals that there exist travelling waves for slow frequency regimes whereas for high frequency case his findings are in accordance with those of Gortler's. Morgan has pointed out that the solutions of rapid forcing problem can be related through a similarity law to the limiting case of irrotational motion. For oscillation of the critical frequency (the parabolic regime) no definite conclusion can be drawn from the linearized equations of motion, but it appears likely that a kind of resonance condition accompanied by violent motion will result.

2. Mathematical Formulation

Let us suppose that the liquid, unlimited in all directions is rotating about Z-axis with uniform angular velocity Ω and that a sphere of radius a oscillates along the axis of rotation with a velocity $V \cos \sigma t$. Further, we assume that at time $t = 0$, the sphere impulsively starts to move with a velocity V. Choosing the origin at the centre of the sphere and adopting cylindrical coordinates (r, θ, z), the equation of the sphere becomes $r^2 + z^2 = a^2$. The perturbation in the velocity of the fluid due to the motion of the sphere is supposed to be sufficiently small so that their squares and products may be neglected. Let the components of the fluid velocity in the perturbed state referred to instantaneously fixed axes along the directions of (r, θ, z) be $(u, \Omega r + v, w)$.

The unsteady motion of the fluid is described by the linearized Euler equation relative to the rotating axes and the equation of continuity, which, in usual notations, are

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - 2\Omega v = \frac{-\partial P}{\partial r} \\ &\frac{\partial v}{\partial t} + 2\Omega u = 0 \; \frac{\partial w}{\partial t} = \frac{-\partial P}{\partial z} \end{aligned}\right\} \qquad \ldots(2.1)$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \qquad \ldots(2.2)$$

where

$$P = \frac{p}{\rho} - \tfrac{1}{2}\Omega^2 r^2.$$

The letters P and ρ represent disturbance pressure and density respectively of the fluid.

Since the motion is symmetrical about the axis of rotation, it is independent of θ.

The boundary conditions are:

$$ru + wz = \operatorname{Re}(Vze^{i\sigma t}) \text{ on the surface of the sphere} \qquad \ldots(2.3a)$$

and $$u, v, w \to 0 \text{ when } z \to \infty \text{ for fixed } r \text{ and } t. \qquad \ldots(2.3b)$$

Taking oscillations for the variables as

$$P = \operatorname{Re}[P'(r, z)\, e^{i\sigma t}],\; u = \operatorname{Re}[u'(r, z)\, e^{i\sigma t}], \text{ etc.} \qquad \ldots(2.4)$$

eqns. (2.1) and (2.2) yield

$$\left.\begin{aligned} &u' = \frac{i\sigma}{\sigma^2 - 4\Omega^2}\frac{\partial P'}{\partial r} \\ &v' = -\frac{2\Omega}{\sigma^2 - 4\Omega^2}\frac{\partial P'}{\partial r},\; w' = -\frac{1}{i\sigma}\cdot\frac{\partial P'}{\partial z} \end{aligned}\right\} \qquad \ldots(2.5)$$

and

$$\frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{\partial w'}{\partial z} = 0. \qquad \text{...(2.6)}$$

Eliminating u', v', w' from (2.5) and using (2.6) the governing equations in P reduce to

$$\frac{\partial^2 P'}{\partial r^2} + \frac{1}{r}\frac{\partial P'}{\partial r} + \left(1 - \frac{4\Omega^2}{\sigma^2}\right)\frac{\partial^2 P'}{\partial z^2} = 0. \qquad \text{...(2.7)}$$

The boundary conditions (2.3a, b) reduce to

$$z\frac{\partial P'}{\partial z} + \frac{r\sigma^2}{(\sigma^2 - 4\Omega^2)}\frac{\partial P'}{\partial r} = -\, i\sigma V z \qquad \text{...(2.8}a\text{)}$$

and

$$\frac{\partial P'}{\partial r}, \frac{\partial P'}{\partial z} \to 0 \text{ as } z \to \infty \text{ for fixed } r \text{ and } t. \qquad \text{...(2.8}b\text{)}$$

Equation (2.7) is elliptic or hyperbolic according as the frequency of oscillation is greater or less than twice the angular velocity of rotation.

3. Elliptic Case ($\sigma > 2\Omega$)

Applying the method of separation of variables to (2.7), we have

$$P' = J_0(kr)\{Ae^{pk} + Be^{-pk}\} \qquad \text{...(3.1)}$$

where $p = \dfrac{z\sigma}{(\sigma^2 - 4\Omega^2)^{1/2}}$ and k is the constant of separation. The condition (2.8b) implies $A = 0$.

$$P' = BJ_0(kr)\exp(-pk).$$

The general solution may be expressed as an integral

$$P' = \int_0^\infty B(k)\, J_0(kr)\exp(-pk)\, dk. \qquad \text{...(3.2)}$$

Hence the surface condition (2.8a) reduces to

$$p\int_0^\infty kB(k)\, J_0(kr)\exp(-pk)\, dk + \frac{r\sigma^2}{\sigma^2 - 4\Omega^2}\int_0^\infty kB(k)\, J_1(kr)\exp(-pk)\, dk$$

$$= i\sigma V z. \qquad \text{...(3.3)}$$

An appropriate solution of (3.3) is

$$B(k) = C\left(\frac{2\pi}{k}\right)^{1/2} J_{1/2}(ka)$$

where C is constant.

Substituting the value of $B(k)$ in (3.3) we have

$$(2\pi)^{1/2}\, Cp \int_0^\infty k^{1/2} J_0(kr)\, J_{1/2}(ak) \exp(-pk)\, dk$$

$$+ \frac{(2\pi)^{1/2}\, Cr\sigma^2}{(\sigma^2 - 4\Omega^2)} \int_0^\infty k^{1/2} J_1(kr)\, J_{1/2}(ak) \exp(-pk)\, dk = i\sigma Vz. \qquad \ldots(3.4)$$

Since

$$\int_0^\infty k^{1/2} J_0(kr)\, J_{1/2}(ak) \exp(-pk)\, dk$$

$$= \frac{(\frac{1}{2}a)^{1/2}\,(2)^{1/2}}{(\frac{3}{2})^{1/2}\,(r^2+p^2)}\; {}_2F_1\left(1, -\tfrac{1}{2}, 1, \frac{r^2}{r^2+p^2}\right)$$

and

$$\int_0^\infty k^{1/2} J_1(kr)\, J_{1/2}(ak) \exp(-pk)\, dk$$

$$= \frac{(\frac{1}{2}a)^{1/2}\,(r/2)\,(3)^{1/2}}{(\frac{3}{2})^{1/2}\,(r^2+p^2)^{3/2}\,(2)^{1/2}}\; {}_2F_1\left(\tfrac{3}{2}, 0, 2, \frac{r^2}{r^2+p^2}\right) p > |a|$$

(Watson 1962, p. 399)

eqn. (3.4) reduces to

$$(2\pi)^{1/2}\, C\left[\left(\frac{2a}{\pi}\right)^{1/2} \frac{p^2}{(r^2+p^2)^{3/2}} + \frac{\sigma^2}{\sigma^2 - 4\Omega^2}\cdot\left(\frac{2a}{\pi}\right)^{1/2}\cdot\frac{r^2}{(r^2+p^2)^{3/2}}\right]$$

$$= i\sigma Vz.$$

Hence

$$C = \frac{i\sigma Vz^3(r^2+p^2)^{3/2}}{2a^{1/2}p^2(z^2+r^2)}$$

and

$$B(k) = \left(\frac{2\pi}{k}\right)^{1/2} \frac{i\sigma Vz^3(r^2+p^2)^{3/2}}{2a^{1/2}p^2(z^2+r^2)}\, J_{1/2}(ak).$$

Therefore

$$P' = (2\pi)^{1/2}\, C \int_0^\infty k^{-1/2} J_0(kr)\, J_{1/2}(ak)\, e^{-pk}\, dk.$$

Also, since

$$\int_0^\infty k^{-1/2} J_0(kr)\, J_{1/2}(ak) \exp(-pk)\, dk$$

$$= \frac{(\frac{1}{2}a)^{1/2}}{(\frac{3}{2})^{1/2}\,(r^2+p^2)^{1/2}}\; {}_2F_1\left(\tfrac{1}{2}, 0, 1, \frac{r^2}{r^2+p^2}\right)$$

$$P' = (2\pi)^{1/2}\,\frac{i\sigma V z^3 (r^2+p^2)^{3/2}}{2a^{1/2} p^2 (z^2+r^2)} \cdot \left(\frac{2a}{\pi}\right)^{1/2} \cdot \frac{1}{(r^2+p^2)^{1/2}}$$

$$= \frac{iVz}{\sigma}\left[\sigma^2 - \frac{4\Omega^2 r^2}{(z^2+r^2)}\right], \quad z > \left|\frac{a(\sigma^2 - 4\Omega^2)^{1/2}}{\sigma}\right|$$

and hence

$$P = Vz\left(\sigma^2 - \frac{4\Omega^2 r^2}{z^2+r^2}\right)\frac{\sin\sigma t}{\sigma}. \qquad \ldots(3.5)$$

Velocity Components

From (2.5), we have

$$u' = -\frac{8V\Omega^2 z^3 r}{(\sigma^2 - 4\Omega^2)(z^2+r^2)^2}$$

$$v' = -\frac{16 V i \Omega^3 z^3 r}{\sigma(\sigma^2 - 4\Omega^2)(z^2+r^2)^2}$$

$$w' = -V\left[1 + \frac{4\Omega^2 r^2 (z^2 - r^2)}{\sigma^2 (z^2+r^2)^2}\right].$$

The velocity components are

$$\left.\begin{aligned} u &= -\frac{8V\Omega^2 z^3 r \cos\sigma t}{(\sigma^2 - 4\Omega^2)(z^2+r^2)^2} \\ v &= -\frac{16 V\Omega^3 z^3 r \sin\sigma t}{\sigma(\sigma^2 - 4\Omega^2)(z^2+r^2)^2} \\ w &= -V\left[1 + \frac{4\Omega^2 r^2 (z^2 - r^2)}{\sigma^2 (z^2+r^2)^2}\right]\cos\sigma t. \end{aligned}\right\} \qquad \ldots(3.6)$$

Thus in the elliptic case there is potential flow and the extent of disturbance decreases as the forcing frequency is reduced or the angular velocity is increased.

Note: When $\sigma = 2\Omega$, u and v breakdown. This implies that physically sensible solution cannot be obtained from the linearized theory. It is thus necessary to include nonlinear and viscous terms in the formulation of the problem.

4. Hyperbolic Case ($2\Omega > \sigma$)

In this case, the governing equation is

$$\frac{\partial^2 P'}{\partial r^2} + \frac{1}{r}\frac{\partial P'}{\partial r} - \left(\frac{4\Omega^2}{\sigma^2} - 1\right)\frac{\partial^2 P'}{\partial z^2} = 0. \qquad \ldots(4.1)$$

Solving it by the method of separation of variables, we have

$$P' = J_0(kr)\,(A \cos p'k + B \sin p'k).$$

where $p' = \dfrac{z\sigma}{(4\Omega^2 - \sigma^2)^{1/2}}$ and k is the constant of separation.

Hence

$$P' = \tfrac{1}{2} J_0(kr)\left[\left(A + \frac{B}{i}\right) e^{ip'k} + \left(A - \frac{B}{i}\right) e^{-ip'k}\right]. \qquad \ldots(4.2)$$

The first and second parts in (4.2) represent incoming and outgoing waves respectively. In view of the Sommerfield's radiation condition (1964) i.e. no free waves are crossing the field from one side to the other or simultaneously coming in from infinity, we find

$$A + B/i = 0 \quad \text{i.e.} \quad B = -Ai.$$

$$P' = AJ_0(kr)\, e^{-ip'k}.$$

and

$$P = \operatorname{Re}\,[AJ_0(kr)\, e^{i(\sigma t - p'k)}]$$

$$= \operatorname{Re}\left[AJ_0(kr) \exp\left\{i\sigma\left(t - \frac{kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)\right\}\right].$$

Hence

$$w = \operatorname{Re}\left[\frac{Ak}{(4\Omega^2 - \sigma^2)^{1/2}} J_0(kr) \exp\left\{i\sigma\left(t - \frac{kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)\right\}\right]. \qquad \ldots(4.3)$$

The nature of wave motion is not readily apparent from axisymmetric solution (4.3). Introducing the asymptotic form (kr large) of Bessel function, we obtain (for the case $z > 0$)

$$w \sim \frac{Ak}{(4\Omega^2 - \sigma^2)^{1/2}} \sqrt{\frac{2}{k\pi r}} \cos\left(kr - \frac{\pi}{4}\right) \cos\left(\sigma t - \frac{\sigma kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)$$

$$\sim \frac{Ak^{1/2}}{\{2\pi r(4\Omega^2 - \sigma^2)\}^{1/2}}\left[\cos\left(\sigma t - \frac{\pi}{4} + kr - \frac{\sigma kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)\right.$$

$$\left. + \cos\left(\sigma t + \frac{\pi}{4} - kr - \frac{\sigma kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)\right].$$

Hence

$$w \sim \frac{Ak^{1/2}}{\{2\pi r(4\Omega^2 - \sigma^2)\}^{1/2}}\left[\cos\left(\sigma T + kr - \frac{\sigma kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)\right.$$

$$\left. + \sin\left(\sigma T - kr - \frac{\sigma kz}{(4\Omega^2 - \sigma^2)^{1/2}}\right)\right],$$

where $\sigma t - \frac{\pi}{4} = \sigma T.$

For large k the motion is a superposition of two wave-trains, their crests lying along the 'characteristic directions' of Gortler's analysis. The wave trains propagate perpendicular to these directions. As the critical frequency $2\Omega = \sigma$, the wave speed drops to zero and the crests lie in the plane $z = 0$.

A general representation of (4.3) can be obtained by introducing an integral representation of the Bessel function

$$J_0(kr) = \frac{1}{\pi} \int_0^{\pi} \cos(kr \sin\theta)\, d\theta.$$

For $z > 0$,

$$w = \frac{Ak}{(4\Omega^2 - \sigma^2)^{1/2}} \frac{1}{2\pi} \left[\int_0^{\pi} 2\cos\left(\sigma t - \frac{\sigma k z}{(4\Omega^2 - \sigma^2)^{1/2}}\right) \cos(kr\sin\theta)\, d\theta \right]$$

$$= \frac{Ak}{2\pi(4\Omega^2 - \sigma^2)^{1/2}} \left[\int_0^{\pi} \left\{ \cos\left(\sigma t - \frac{\sigma k z}{(4\Omega^2 - \sigma^2)^{1/2}} + kr\sin\theta\right) \right.\right.$$

$$\left.\left. + \cos\left(\sigma t - \frac{\sigma k z}{(4\Omega^2 - \sigma^2)^{1/2}} - kr\sin\theta\right) \right\} d\theta \right],$$

the asymptotic solution being represented as a superposition of two-dimensional waves whose crests have slopes $\frac{(4\Omega^2 - \sigma^2)^{1/2}}{\sigma} \sin\theta$, smaller than or equal to those predominating at large kr.

5. Conclusion

If we compare the forms of solution in the elliptic and in the hyperbolic cases we see that in the former the disturbance tends to zero with increasing z whereas in the latter it propagates throughout the fluid.

When $\sigma = 2\Omega$, u and v become indefinitely large and hence no definite conclusion can be drawn from the linearized equations. This fact combined with the observed behaviour of the elliptic and hyperbolic solution as $\sigma \to 2\Omega$ may be interpreted that for oscillations of a critical frequency $\sigma = 2\Omega$ a kind of resonance condition may be set up which results a violent motion.

These results correspond to those obtained by Morgan and Reynolds.

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