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FLOW OF NON-NEWTONIAN FLUID IN A HELICAL PIPE WITH AN ELLIPTICAL CROSS-SECTION

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A theoretical analysis has been carried out of the flow of an incompressible non-Newtonian viscous fluid in a helical pipe; this type of pipes is characterized by non-zero pitch. The motion equations are derived, and it is shown that they are depend on Dean number L, the non-Newtonian parameter β and the parameter λ/Re where λ is the ratio of the torsion τ to the curvature κ of the pipe axis and Re is Reynolds number. The solutions of motion equations are expanded in term of Dean number. Analytical solutions of perturbation equations are derived for small values of Reynolds number by using Galerkin variation method. The effect of the parameters mentioned above are studied upon the secondary flow and the axial velocity. Also we improve and recover many subcases, which have been studied before, such as, the flow of Newtonian fluid in a helical pipe with elliptical cross-section, the flow of Newtonian and non-Newtonian fluid in a curved pipe with elliptical cross-section and the flow in a straight pipe.

Key Words : Fluid Mechanics; Non-Newtonian Fluid; Helical Pipe

1. INTRODUCTION

The researchers on helical pipes (non-zero pitch) with various cross-sections started to appear recently. To the author's knowledge, no theoretical work has been done on the flow of non-Newtonian liquids through helical pipes. The survey of previous literature that is given below is about the flow of Newtonian fluids through helical pipes and most of these cases are recovered through out this paper.

The flow of Newtonian fluid has been the subject of extensive theoretical studies till date. Dean[3] has been the first researcher who worked in flow analysis of Newtonian fluids in curved pipe. He introduced a toroidal coordinate system to show that the relation between pressure gradient and the rate of flow through a curved pipe with circular cross-section of incompressible Newtonian fluid is dependent on the curvature. In that paper he could not show this dependence but did in his second paper,[4] he modified his analysis by including the higher order terms and was able to show that the rate of flow is slightly reduced by curvature.

Wang[10], introduced a non-orthogonal helical coordinate system to study the effect of the curvature and the torsion on the flow of incompressible Newtonian fluid in a helical pipe with circular cross-section and employed a series expansion method in term of $\varepsilon = \kappa a << 1$, where κ is the curvature and a is the radius of the pipe. He found a first order effect of torsion on the secondary flow, which consisted of two non-symmetric vortexes.

Germano[5], in (1982) has used an orthogonal coordinate system to study the effect of the torsion on the Newtonian fluid flow in helical pipe with circular cross-section, he found only second order effect of torsion on the secondary mtion and this result is confirmed in his second paper[6] in

which he studied the effect of torsion on a helical pipe with an elliptical cross-section and showed that there is an unexpected form of the secondary flow where the walls act as sources and sinks.

Finally, Bolinder[2], in (1996) employed a series expansion method to determine the first order terms in curvature ε and torsion η of fully developed laminar flow in a helical square duct and in a helical rectangular duct of aspect ratio two. His main conclusions are that the flow in a helical duct with a finite torsion is obtained as a superposition of the flow in a toroidal duct with zero torsion and a straight twisted duct; that the secondary flow in helical non-circular ducts for sufficiently small Reynolds number dominates torsion effects; and that increasing Reynolds number, the secondary flow eventually is dominated by effects due to the curvature.

The methods of solution given by Germano and Bolinder or in a more accurate statement the expansions, which they have used, are equivalent. But the researcher can see that the expansion that is given by Germano is better because it reduces the number of the equations that are needed to be solved in order to analyze the fluid motion.

In this paper, we are going to deal with the case of viscous, incompressible, second order fluid flowing in a helical pipe with an elliptical cross-section under the action of the pressure gradient. In particular, the consideration is given to visco-inelastic liquid, such as polymers, tar ... etc, which can be represented by equation of state of the form

$$T_{ik} = 2\,\eta\, e_{ik} + 4\,\xi\, e_{ij}\, e_{jk}. \qquad \ldots (1)$$

Here η is the viscosity coefficient and ξ is the normal stress coefficient and they are assumed to be constants for the fluid to be experimentally valid. Also T_{ij} and e_{ij} are the stress and the rate of strain tensors respectively[9].

2. COORDINATES SYSTEM

In the study of the flow of non-Newtonian fluid in a helical pipe, it is very important to choose appropriate coordinate system so that the resulting motion equations can be dealt with conveniently. In such fluids the equations are more complicated than the case of Newtonian fluid. It is found that the orthogonal coordinate system, which is introduced by Mercier and used by Germano[6], is the appropriate one among the systems that we have examined.

Let the position vector be given by

$$\boldsymbol{R}(s) = \boldsymbol{X}(s)\,\boldsymbol{i} + \boldsymbol{Y}(s)\,\boldsymbol{j} + \boldsymbol{Z}(s)\,\boldsymbol{k}, \qquad \ldots(2)$$

where s is the arc length along the pipe and $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{i}$ are unites vectors in Cartesian direction. The ***TNB*** frame and Frenet formulas are defined by

$$\left.\begin{aligned} \boldsymbol{T} = \frac{d\boldsymbol{R}}{ds},\ \boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds},\ \boldsymbol{B} = \boldsymbol{T}\times\boldsymbol{N} \\ \frac{d\boldsymbol{N}}{ds} = \tau\boldsymbol{B} - \kappa\boldsymbol{T},\ \frac{d\boldsymbol{B}}{ds} = -\tau\boldsymbol{N} \end{aligned}\right\}. \qquad \ldots (3)$$

Here $\boldsymbol{T}, \boldsymbol{N}$ and $\boldsymbol{B}$ are the tangent, normal and binormal respectively, τ is the torsion and κ is the curvature. To construct the orthogonal coordinate system (s, r, θ) let the polar angle θ is referred to a rotation of unit vector $\boldsymbol{N}^*$ by a mount of $\phi + \phi_0$, where ϕ is given by

$$\phi = -\int_{s_0}^{s} \tau(s)\, ds'.$$

And ϕ_0 is an arbitrary constant angle, [Fig. 1], we have

$$\boldsymbol{x} = \boldsymbol{R}(s) + r$$

$$r = -\cos(\pi - (\theta + \phi + \phi_0)\boldsymbol{N}(s) + \sin(\pi - (\theta + \phi + \phi_0)\boldsymbol{B}(s).$$

Then we can write

$$\boldsymbol{x} = P - O = \boldsymbol{R}(s) + r\cos(\theta + \phi + \phi_0 \boldsymbol{N}(s) + r\sin(\theta + \phi + \phi_0)\boldsymbol{B}(s) \quad \ldots (4)$$

It is easy to prove that

$$d\boldsymbol{x} = (1 + \kappa r \sin(\theta + \phi))\, ds\, e_s + dr\, e_r + rd\,\theta\, e_\theta \quad \ldots (5)$$

and

$$d\boldsymbol{x} \cdot d\boldsymbol{x} = (1 + \kappa r \sin(\theta + \phi))^2 (ds^2) + (dr)^2 + r^2 (d\theta)^2 \quad \ldots (6)$$

where

$$e_s = \boldsymbol{T}$$

$$e_r = \boldsymbol{B}\cos(\theta + \phi) - \boldsymbol{N}\sin(\theta + \phi)$$

and

$$e_\theta = -\boldsymbol{B}\sin(\theta + \phi) - \boldsymbol{N}\cos(\theta + \phi)$$

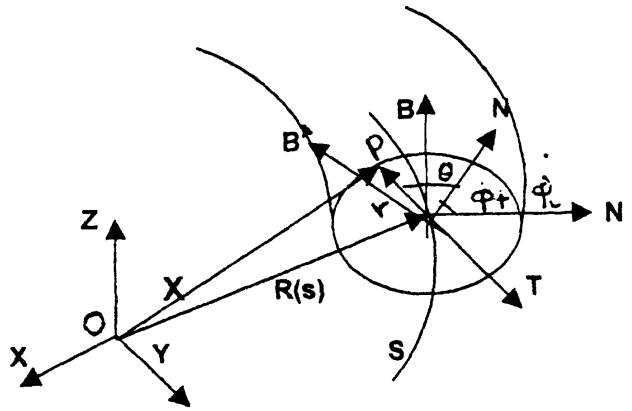

FIG. 1. Coordinate system

and $\phi_0 = \pi/2$. It is clear that from the metric given by (6) that this coordinate system is orthogonal.

3. MOTION AND CONTINUITY EQUATIONS

Since our coordinates are orthogonal, it is possible to use orthogonal curvilinear coordinates system to derive the motion equations. The explicit form of rate-of-strain, stress, motion equations and continuity equation can be found in [1 & 8]. In our case, from eq. (6), we have

$$\left.\begin{array}{lll} h_1 = \dfrac{1}{1+\kappa r \sin(\theta+\phi))}, & h_2 = 1, h_3 = \dfrac{1}{r} \\ \alpha = s & , \beta = r\ , & \gamma = \theta \\ u_\alpha = u & , u_\beta = v, & u_\gamma = w \end{array}\right\} \quad \ldots (7)$$

Here α, β, γ are the curvilinear coordinates. If u, v and w are the velocity components in the coordinate direction, then it is easy to obtain the dimensional form of motion and continuity equations*. These equations can be written in non-dimensional form through the following relations

$$u = U_0 u_1, \nu = \frac{\nu}{a}\nu_1, w = \frac{\nu}{a} w_1, s = as_1, r = ar_1,$$

$$\lambda = \tau/\kappa, \varepsilon\, \kappa a, p = U_0^2 p_1, \mathrm{Re} = \frac{U_0 a}{\nu}$$

where a and U_0 are the radius of the pipe and the maximum velocity in a straight pipe under the pressure gradient respectively, ν is the viscosity and p_1 is the kinematic pressure define by

$$p_1 = p_0(s) + \varepsilon p_{11}(s_1, r_1) = -\frac{G}{\mathrm{Re}} s + \varepsilon p_{11}(s_1, r_1),$$

where G is constant given by $[(Re\ a/\rho\, U_0^2)\, P^{**}]\, P^{**}$ is the pressure gradient. And in order to obtain helically symmetric solution of the general equations, which is physically corresponding to a fully developed flow in a helical pipe and can be operated by the transformations

$$\theta + \phi \Rightarrow \theta_1, \frac{\partial}{\partial s} \Rightarrow \frac{\partial}{\partial s} - \varepsilon\lambda \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta_1}.$$

And setting all the resulting derivatives with respect to s equal to zero except the pressure derivative.

In what follow we shall omit the indices, it being understood that all the variables are dimensionless. And for mathematical convenient, consideration is given to a helical pipe with constant curvature κ and torsion τ.

3.1 Motion Equations in Stream Function Form

The non-dimensional equations may be obtained now by introducing pseudo-stream function ψ which is define by

$$\nu = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad w = \frac{\partial \psi}{\partial r} + \frac{\lambda}{2\,Re} Lru, \quad \ldots (8)$$

where $L = \varepsilon Re^2$ is Dean number, and these non-dimensional equations of motion, which define the flow of non-Newtonian fluid in a helical pipe, are :

*See A. M. Hadi "Flow analysis through curved pipes" *Ph.D Thesis*, Pune Univ. Dept. of Math.

$$\nabla^2 u + G = H_1\left(r, \theta, \beta, \beta L\left(\frac{\lambda}{Re}\right)\right), \qquad \ldots (9)$$

$$\nabla^4 \psi = H_2\left(r, \theta, L, \beta L\left(\frac{\lambda}{Re}\right), \beta L^2\left(\frac{\lambda}{Re}\right)^2\right), \qquad \ldots (10)$$

and the continuity equation which is satisfy by pseudo-stream function (8) is

$$\frac{\partial (r v)}{\partial r} + \frac{\partial}{\partial \theta}\left(w - \frac{\lambda}{2Re} Lru\right) = 0 \qquad \ldots (11)$$

where $\beta = \xi/\rho a^2$ is the non-Newtonian parameter. The motion eqs. (9) and (10) are controlled by three parameters namely, Dean number L, the parameter λ/Re and the non-Newtonian parameter β.

It is easy to see that if $\beta = 0$ these equations reduce to the corresponding system in the case of Newtonian fluid (Germano equations, [6]).

We shall easy to see that if $\beta = 0$ these equations reduce to the corresponding system in the case of Newtonian fluid (Germano equations, [6]).

We shall study the effect of the three parameters mentioned above on the secondary flow and the axial velocity when the cross-section is ellipse, and as special case when the cross-section is circular, also we will recover the cases of Newtonian fluid in a curved pipe and helical pipes as a matter of comparison.

4. METHOD OF SOLUTION

Let us first consider the flow of non-Newtonian fluid, characterized by (1), in a helical pipe with an elliptical cross-section whose semiaxes and majoraxes are of length $2a$, $2b$ ($b > a$) respectively and oriented in the direction of the binormal B and normal N to the helical axes, see Fig. 2.

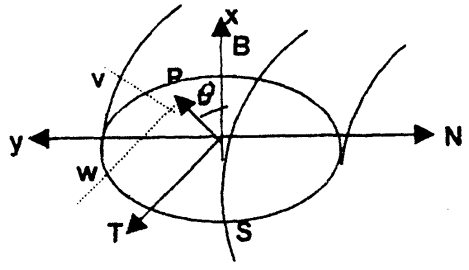

FIG. 2. Elliptical cross-section with axes oriented with N and B

The equation of the boundary of the pipe, referred to Cartesian axes x and y is taken to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Expressed in polar coordinates, it is written as

$$1 - r^2 + (1 - c^2) r^2 \sin\theta = 0, \qquad \ldots (12)$$

where r is dimensionless with respect to (a), $c = a/b$ is the aspect ratio and the non-slip conditions are

$$u = \psi = \frac{\partial \psi}{\partial r} = 0 \text{ at } r = \frac{1}{\sqrt{1-(1-c^2)\sin(\theta)}}. \qquad \ldots (13)$$

The solutions of eqs. (9) and (10) subject to associated boundary conditions are $\psi(r, \theta, L, \lambda/Re, \beta)$, $u(r, \theta, L, \lambda/Re\,\beta)$, the prime parameter is Dean number L, and successive approximation method is adopted. This successive method is equivalent to expand ψ and u in ascending power of Dean number L. Thus we have

$$\left.\begin{aligned} \psi &= L\,\psi_1 + L^2\,\psi_2 + \ldots \\ u &= u_0 + L u_1 + L^2 u_2 + \ldots \end{aligned}\right\}. \qquad \ldots (14)$$

4.1 SUCCESSIVE APPROXIMATION

If we substitute the above expressions for ψ and u in (9) and (10) and equating coefficient of equal power in L, we obtain recursive relations from which $u_0, \psi_1, u_1, \psi_2, \ldots$ can be successively determined. The relations are :

$$\nabla^2 u_0 = -G, \qquad \ldots (15)$$

$$\nabla^4 \psi_1 = u_0\left(\frac{\sin\theta}{r}\frac{\partial u_0}{\partial \theta} - \cos\theta\frac{\partial u_0}{\partial r}\right) + 2G\frac{\lambda}{Re}, \qquad \ldots (16)$$

and

$$\nabla^2 u_1 = \frac{1}{r}\left(\frac{\partial \psi_1}{\partial r}\frac{\partial u_0}{\partial \theta} - \frac{\partial \psi_1}{\partial \theta}\frac{\partial u_0}{\partial r}\right) - \beta\left(\frac{\lambda}{Re}\right)$$

$$\left[\frac{1}{2r}\left(\frac{\partial u_0}{\partial \theta}\frac{\partial u_0}{\partial r} + \frac{3}{2}\frac{\partial u_0}{\partial r}\frac{\partial^2 u_0}{\partial \theta \partial r} + \frac{1}{2}\frac{\partial u_0}{\partial \theta}\right.\right.$$

$$\left.\left.\frac{\partial^2 u_0}{\partial r^2} + \frac{2}{r^2}\frac{\partial u_0}{\partial \theta}\frac{\partial^2 u_0}{\partial \theta^2}\right] - \beta\right.$$

$$\left[\frac{2}{r^2}\frac{\partial^2 u_0}{\partial r^2}\frac{\partial \psi_1}{\partial \theta} - \frac{2}{r}\frac{\partial^2 u_0}{\partial r^2}\frac{\partial^2 \psi_1}{\partial \theta \partial r} - \frac{2}{r^3}\frac{\partial u_0}{\partial r}\frac{\partial \psi_1}{\partial \theta}\right.$$

$$+ \frac{1}{r^2}\frac{\partial u_0}{\partial r}\frac{\partial^2 \psi_1}{\partial \theta \partial r} - \frac{1}{r}\frac{\partial u_0}{\partial r}\frac{\partial^3 \psi_1}{\partial \theta \partial r^2} - \frac{1}{r^2}\frac{\partial u_0}{\partial \theta}\frac{\partial^2 \psi_1}{\partial r^2}$$

$$\left. + \frac{2}{r}\frac{\partial^2 u_0}{\partial \theta \partial r}\frac{\partial^2 \psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial \theta}\frac{\partial^3 \psi_1}{\partial r^3}\right.$$

$$+\frac{1}{r^3}\frac{\partial u_0}{\partial\theta}\frac{\partial\psi_1}{\partial r}-\frac{2}{r^2}\frac{\partial^2 u_0}{\partial\theta\partial r}\frac{\partial\psi_1}{\partial r}-\frac{2}{r^2}\frac{\partial^2 u_0}{\partial\theta\partial r}\frac{\partial^2\psi_1}{\partial\theta^2}$$

$$+\frac{1}{r^3}\frac{\partial u_0}{\partial\theta}\frac{\partial^3\psi_1}{\partial\theta^2\partial r}-\frac{1}{r^3}\frac{\partial u_0}{\partial r}$$

$$\left.\frac{\partial^3\psi_1}{\partial\theta^3}+\frac{2}{r^3}\frac{\partial^2 u_0}{\partial\theta^2}\frac{\partial^2\psi_1}{\partial\theta\partial r}-\frac{2}{r^4}\frac{\partial^2 u_0}{\partial\theta^2}\frac{\partial\psi_1}{\partial\theta}\right] \quad \ldots (17)$$

The above equations are up to first order expansion in L, it is noted that up to this order the stream function ψ does not depend on the non-Newtonian parameter. In order to study the effect of this parameter upon the secondary and axial flow we should go to the next order approximation for ψ, and this is given by

$$\nabla^4\psi_2=\frac{1}{r}\left(\frac{\partial\psi_1}{\partial r}\frac{\partial}{\partial\theta}-\frac{\partial\psi_1}{\partial\theta}\frac{\partial}{\partial r}\right)\nabla^2\psi_1+u_0$$

$$\left(\frac{\sin\theta}{r}\frac{\partial u_1}{\partial\theta}-\cos\theta\frac{\partial u_1}{\partial r}\right)+u_1\left(\frac{\sin\theta}{r}\frac{\partial u_0}{\partial\theta}-\cos\theta\frac{\partial u_0}{\partial r}\right)$$

$$+\beta\left(\frac{\lambda}{Re}\right)\left[-\frac{2}{r}\frac{\partial u_0}{\partial\theta}\frac{\partial^3\psi_1}{\partial r^3}-2\frac{\partial^2 u_0}{\partial\theta\partial r}\frac{\partial^3\psi_1}{\partial r^3}-\frac{1}{r}\frac{\partial^2 u_0}{\partial\theta\partial r}\right.$$

$$\frac{\partial^2\psi_1}{\partial r^2}-\frac{\partial^3 u_0}{\partial\theta\partial r^2}\frac{\partial^2\psi_1}{\partial r^2}+\frac{1}{r^2}\frac{\partial u_0}{\partial\theta}$$

$$\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r^3}\frac{\partial u_0}{\partial\theta}\frac{\partial\psi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_0}{\partial\theta\partial r}\frac{\partial\psi_1}{\partial r}+\frac{1}{r}\frac{\partial^3 u_0}{\partial\theta\partial r^2}$$

$$-\frac{4}{r^4}\frac{\partial u_0}{\partial\theta}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{3}{r^3}\frac{\partial^2 u_0}{\partial\theta\partial r}$$

$$\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial^3 u_0}{\partial\theta\partial r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{2}{r^3}\frac{\partial u_0}{\partial\theta}$$

$$\frac{\partial^3\psi_1}{\partial r\partial\theta^2}-\frac{4}{r^4}\frac{\partial^2 u_0}{\partial\theta^2}\frac{\partial\psi_1}{\partial\theta}+\frac{4}{r^3}\frac{\partial^3 u_0}{\partial r\partial\theta^2}\frac{\partial\psi_1}{\partial\theta}$$

$$\left.+\frac{2}{r^3}\frac{\partial^2 u_0}{\partial\theta^2}\frac{\partial^2\psi_1}{\partial\theta\partial r}-\frac{4}{r^2}\frac{\partial^3 u_0}{\partial\,\partial\theta^2}\frac{\partial^2\psi_1}{\partial\theta\partial r}-\frac{2}{r^2}\frac{\partial u_0}{\partial\theta}\frac{\partial^2 u_1}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_1}{\partial\theta}\right.$$

$$\frac{\partial^2 u_0}{\partial \theta^2} - \frac{3}{2r}\frac{\partial u_0}{\partial \theta}\frac{\partial^3 u_1}{\partial r \partial \theta^2} - \frac{3}{2r}\frac{\partial u_1}{\partial \theta}\frac{\partial^3 u_0}{\partial r \partial \theta^2} - \frac{2}{r^2}\frac{\partial^2 u_0}{\partial \theta^2}\frac{\partial^3 \psi_1}{\partial \theta \partial r^2}$$

$$- \frac{2}{r^2}\frac{\partial^2 u_0}{\partial \theta \partial r}\frac{\partial^3 \psi_1}{\partial r \partial \theta^2}$$

$$+ \frac{1}{r^2}\frac{\partial^3 u_0}{\partial \theta^3}\frac{\partial^2 \psi_1}{\partial r^2} - \frac{1}{r^3}\frac{\partial^3 u_0}{\partial \theta^3}\frac{\partial \psi_1}{\partial r} - \frac{2}{r^4}\frac{\partial^2 u_0}{\partial \theta^2}\frac{\partial^3 \psi_1}{\partial \theta^3}$$

$$- \frac{1}{r^4}\frac{\partial^3 u_0}{\partial \theta^3}\frac{\partial^2 \psi_1}{\partial \theta^2} - \frac{1}{r^4}\frac{\partial u_0}{\partial \theta}\frac{\partial^4 \psi_1}{\partial \theta^4}$$

$$- \frac{\partial u_0}{\partial \theta}\frac{\partial^4 \psi_1}{\partial r^4}\Bigg] + \beta\left(\frac{\lambda}{\mathrm{Re}}\right)^2\Bigg[- \frac{3}{2r}\frac{\partial u_0}{\partial \theta}\frac{\partial u_0}{\partial r} - \frac{5}{2r}\frac{\partial u_0}{\partial \theta}\frac{\partial^2 u_0}{\partial r^2}$$

$$- \frac{5}{2r}\frac{\partial^2 u_0}{\partial \theta \partial r}\frac{\partial_0}{\partial r} - r\frac{\partial^2 u_0}{\partial \theta \partial r}\frac{\partial^2 u_0}{\partial r^2} -$$

$$u_0\frac{\partial^3 u_0}{\partial \theta \partial r^2} - r\frac{\partial u_0}{\partial r}\frac{\partial^3 u_0}{\partial \theta \partial r^2} - \frac{1}{r}\frac{\partial^2 u_0}{\partial \theta \partial r}\frac{\partial^2 u_0}{\partial \theta^2}$$

$$+ \frac{1}{2r}\frac{\partial u_0}{\partial r}\frac{\partial^3 u_0}{\partial \theta^3} - \frac{1}{2}ru_0\frac{\partial^3 u_0}{\partial r^3}\Bigg]. \qquad \text{... (18)}$$

Again, in the above system if $\beta = 0$ this system reduce to the corresponding system in case of Newtonian fluid flow in a helical pipe, (Germano, eqs. 11 and 12[6]), and if we set the aspect ratio $c = 1$ we cover the non-Newtonian fluid flow in a helical pipe with circular cross-section which will given in detail in feature paper.

4.2. Galerkin Solutions of Successive Aproximation Functions

In order to solve eqs. (15-18) subject to associated boundary conditions (13), we employed Galerkin variation method[7]

For eq. (15), assume the solution to be of the form

$$u_0 = a_0 (1 - r^2 + (1 - c^2) r^2 \sin \theta),$$

where a_0 is a constant to be determined. Using the procedure mentioned above, a solution for straight pipe equation is found to be

$$u_0 = 1 - r^2 + (1 - c^2) r^2 \sin \theta, \qquad \text{... (19)}$$

provided that

$$G = 4 - 2(1 - c^2). \qquad \ldots (20)$$

On substituting the expressions for u_0 and G from (19) and (20) into (16) we obtain

$$\nabla^4 \psi_1 = 2r\cos\theta - 2r^3\cos\theta + 2(1-c^2)\, r^3 \sin^2\theta\cos\theta + 4(1-c^2)\frac{\lambda}{Re}. \qquad \ldots (21)$$

It is found that the solution of (21) is given by

$$\psi_1 = \psi_{11} + h\left(\frac{\lambda}{Re}\right)\psi_{12} \qquad \ldots (22)$$

where

$$\left.\begin{aligned} \psi_{11} &= u_0^2 (a_1 r\cos\theta + a_2 r^3\cos\theta + a_3 r^3\cos^3\theta) \\ \psi_{12} &= u_0^2 \end{aligned}\right\} \qquad \ldots (23)$$

And a_1, a_2, a_3, h are functions of the aspect ratio.

It is easy to see that if $c = 1$ and $\lambda = 0$ in (23), this solution reduce to first order Dean solution (eq. 24, [4]).

A solution for u_1 can be obtaining by substituting the expressions for u_0 and ψ_1 from (19) and (22) respectively into (17). The solution for u_1 is given by

$$\begin{aligned} u_1 = (1 - r^2 + c^2 r^2 \sin^2\theta)\,[&(b_1 r\sin\theta + b_2 r^3\sin\theta \\ &+ b_3 r^3\sin^3\theta + b_4 r^5\sin\theta + b_5 r^5\sin^3\theta \\ &+ b_6 r^7\sin\theta + b_7 r^7\sin^3\theta + b_8 r^5\sin^5\theta + b_9 r^7\sin^5\theta + b_{10} r^7\sin^7\theta) \\ &- \beta\left(\frac{\lambda}{Re}\right)(br^2\sin\theta\cos\theta) - \beta(d_1 r\sin\theta + d_2 r^3\sin\theta + d_3 r^5\sin\theta \\ &+ d_4 r^3\sin^3\theta + d_5 r^5\sin^3\theta + d_6 r^5\sin^5\theta)] \end{aligned} \qquad \ldots (24)$$

where $b_1, b_2, \ldots, b_5, b_6$ are long algebraic function of a_1, a_2, a_3 and the aspect ratio c.

The last equation shows that u_1 is depended on (λ/Re) and β. This is the first contribution of the non-Newtonian parameter to the velocity field, while the corresponding equation for Newtonian fluid ($\beta = 0$) is independ of (λ/Re).

Similarly, if we substitute the expressions for ψ_1, u_0, u_1 into (18), a solution for ψ_2 that satisfies the boundary conditions, for $c = 0.65$, is found to be of the form

$$\psi_2 = F_1 + \left(\frac{\lambda}{Re}\right)F^2 + \left(\frac{\lambda}{Re}\right)^2 F_3 + \beta F_4 + \beta\left(\frac{\lambda}{Re}\right)F_5$$

$$+\beta\left(\frac{\lambda}{Re}\right)^2 F_6+\beta^2\left(\frac{\lambda}{Re}\right)F_7+\beta^2\left(\frac{\lambda}{Re}\right)^2 F_8, \qquad \text{... (25)}$$

where $F_1, F_2 \ldots F_8$ are functions of (r, θ).

5. Components of the Secondary Flow

From eq. (25) it can see that ψ is depended on the non-Newtonian parameter β. In (1996), Bolinder proved that the pseudo-stream function which is defined by (8) does not represent a stream function but a stream tubes. So, in order to study the effect of the three parameters L, (λ, Re) and β upon the secondary flow, it is necessary to calculate the components of the secondary flow, which can be obtained by substituting for u_0, ψ_1, u_1 and ψ_2 from (19), (22), (24) and (25) respectively into (8). We further define

$$\left.\begin{aligned} v_x &= \frac{v}{a}(v\cos\theta\, w\sin\theta) \\ v_y &= \frac{v}{a}(v\sin\theta + w\cos\theta) \end{aligned}\right\} \qquad \text{... (26)}$$

where v_x, v_y are dimensionless with respect to v/a. Now eq. (26) takes the form

$$v_x = 2\,\varepsilon\, U_0\, Re\left[\left\{-\frac{1}{r}\left(\frac{\partial\psi_{11}}{\partial\theta}+h\frac{\lambda}{Re}\frac{\partial\psi_{12}}{\partial\theta}\right)-\frac{1}{r}L\frac{\partial\psi_2}{\partial\theta}\right\}\cos\theta-\left\{\left(\frac{\partial\psi_{11}}{\partial r}+h\frac{\lambda}{Re}\frac{\partial\psi_{12}}{\partial r}+\frac{\lambda}{2\,Re}ru+\frac{\partial\psi_2}{\partial r}\right\}\sin\theta\right] \qquad \text{... (27}a\text{)}$$

and

$$v_y = 2\,\varepsilon\, U_0\, Re\left[\left\{-\frac{1}{r}\left(\frac{\partial\psi_{11}}{\partial\theta}+h\frac{\lambda}{Re}\frac{\partial\psi_{12}}{\partial\theta}\right)-\frac{1}{r}L\frac{\partial\psi_2}{\partial\theta}\right\}\sin\theta-\left\{\left(\frac{\partial\psi_{11}}{\partial r}+h\frac{\lambda}{Re}\frac{\partial\psi_{12}}{\partial r}+\frac{\lambda}{2\,Re}ru+L\frac{\partial\psi_2}{\partial r}\right\}\cos\theta\right] \qquad \text{... (27}b\text{)}$$

6. Results and Discussion

In this section we will study the effect of the parameters L, (λ, Re) and the non-Newtonian parameter β upon the components of the secondary flow and the axial velocity both, the second equation of (14), in case of an elliptical and circular cross-section. Since the pseudo-stream function, eq. (8) for a duct with non-zero torsion does not represent the secondary flow as described by v and w, therefore vector plots are employed to present the secondary flow.

6.1. SECONDARY FLOW MOTION

A helical pipe characterized by non-zero torsion, i.e. (λ/Re) is greater than zero; when (λ/Re) is zero the helical pipe reduce to curved pipe and when $(\lambda/\mathrm{Re}) \to \infty$ it reduce to the straight pipe. We have plotted more than 25 cases to ascertain how the parameter (λ/Re), Dean number L and the non-Newtonian parameter β affect the secondary flow in helical, curved and straight pipes. One of the difference between a flow in a curved pipe and a flow in helical pipe is that there is two symmetrical vortices appear in the cross-section of the curved pipe, while these vortices are not symmetric in case of helical pipe and this is due to non-zero torsion. Let us now consider a helical pipe with $a = 0.65$ and $b = 1$ $(c < 1)$.

Fig. (3) shows the effect of the non-Newtonian parameter β on the secondary flow. For $\lambda/Re = 0.02$ $(Re = 10)$, it is observed that as β increases from 0.01 to 0.3 there is displacement in the center of the upper vortex towards the outside of the bend. In addition to this, the upper

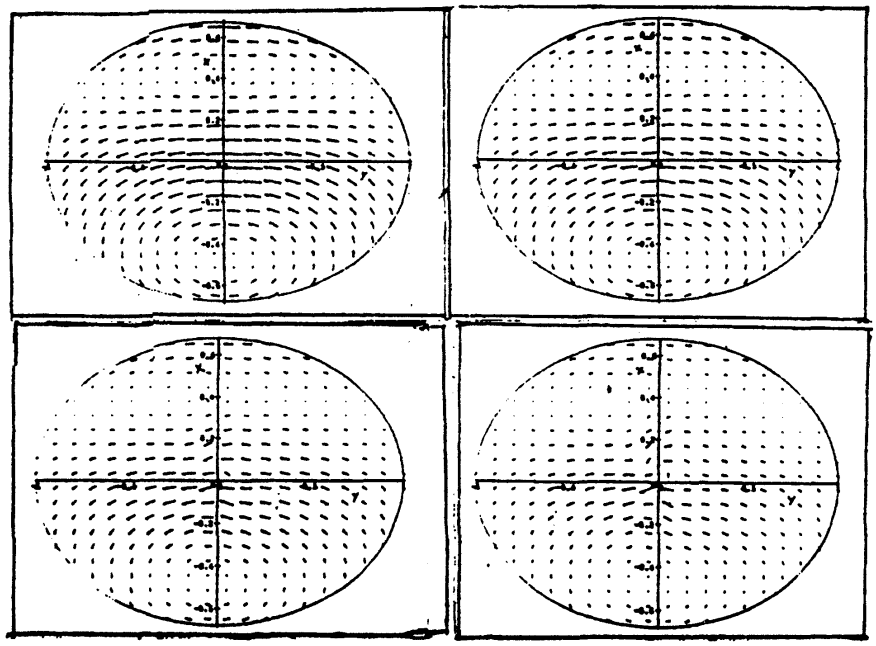

FIG. 3. Secondary flow components for $(\lambda/\mathcal{R}e) = 0.02$, $\mathcal{R}e = 10$. (a) $\beta = 0.01$, (b) $\beta = 0.08$, (c) $\beta = 0.15$, (d) β 0.03.

vortex become smaller and lower one becomes bigger, this means that the intensity of the secondary flow in the lower part of the cross-section is getting weaker and the intensity in the upper part is getting stronger, see Figs. (3*a*, 3*b*, 3*c* & 3*d*).

Fig. (4) illustrates the effect of the torsion on the component of the secondary flow. Here Reynolds number was set to 10 (Dean number 13) and the non-Newtonian parameter β was 0.05 and λ ($\lambda = \tau/\kappa$, for fixed κ) varied from 0.01 to 0.1, it i noted that as τ increases, the two-celled structure gradually disappears, see figures (4*a*, 4*b*, 4*c*, 4*d*, 4*e*); when $\tau = 0.04$, figure 4*d*, we see that the outside of the bend apparently act as a source and the inside of the bend as a sink. The rotating torsion of the wall pushes the flow in the direction of torsion. This phenomenon looks clearer when Reynolds number decreases, if we refer to Fig. (4f) we see that this is the cases when $Re = 2$ and $\tau = 0.02$, which means that the effect of Reynolds number is similar to the effect of the torsion.

In eq. (27), if $\beta = 0$ we recover the first order result in L of Germano[6], for Newtonian fluid flow in a helical pipe with an elliptical cross-section, Fig. (5) illustrate the effect of the

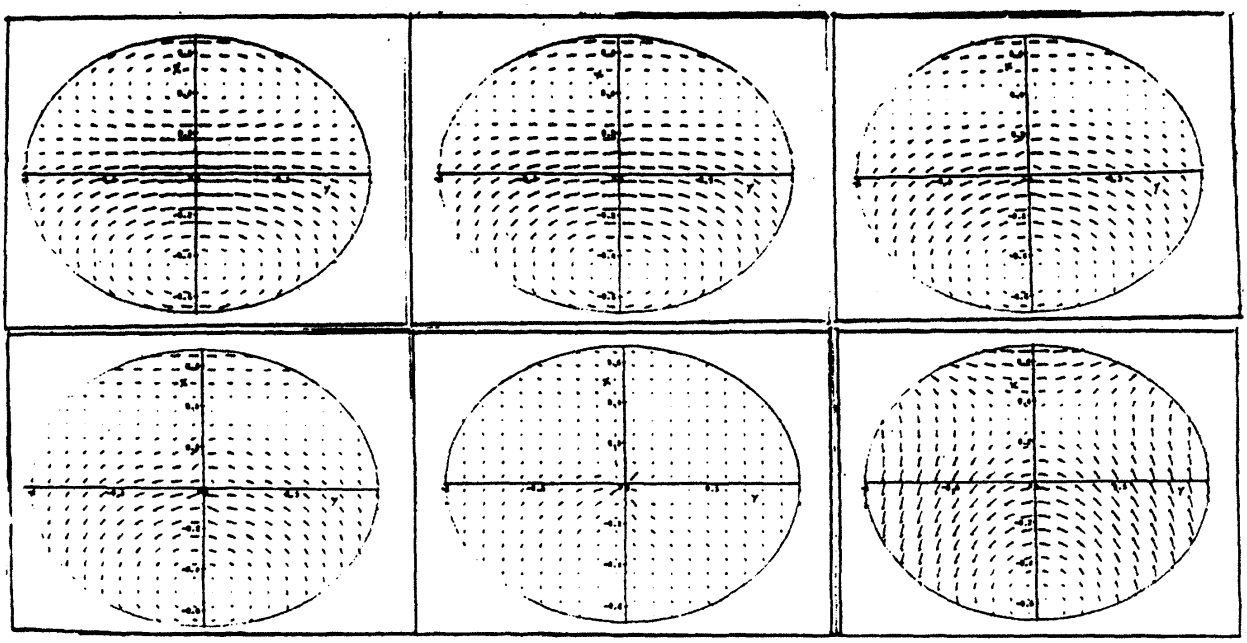

FIG. 4. Secondary flow components for $\beta = 0.05$, $\mathcal{R}e = 10$. (a) $\tau = 0.01$, (b) $\tau = 0.02$, (c) $\tau = 0.03$, (d) $\tau = 0.04$, (e) $\tau = 0.01$, (f) $\tau = 0.02$, $\mathcal{R}e = 2$.

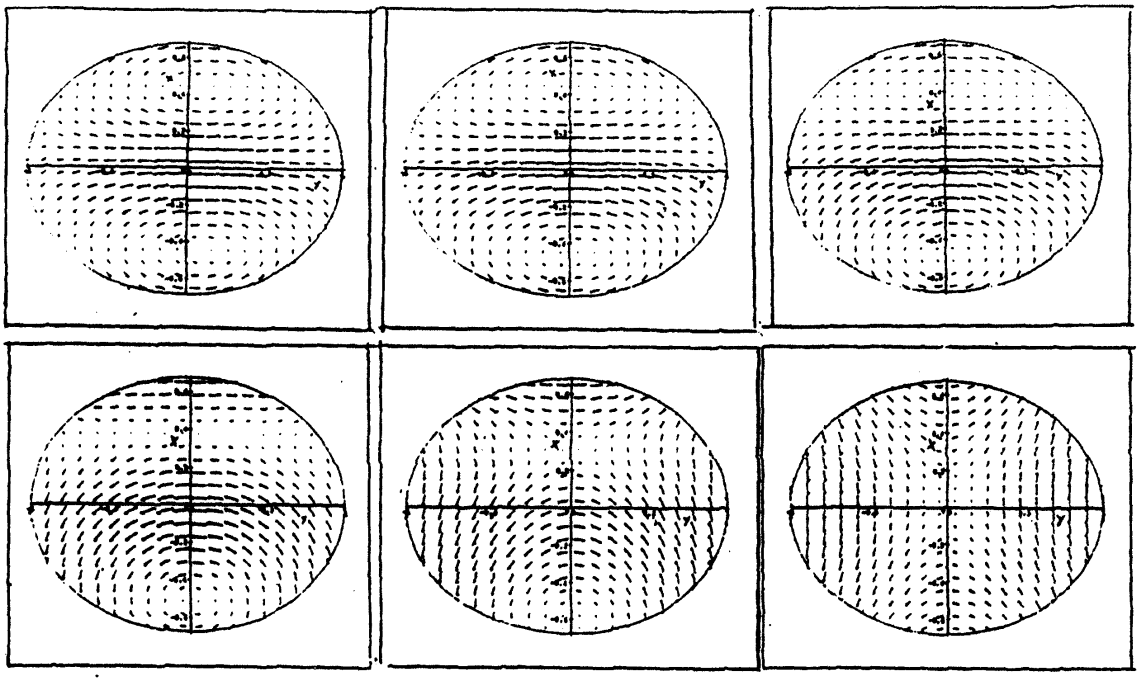

FIG. 5. Secondary flow components for $c = 0.65$. (a) $(\lambda/\mathcal{R}e) = 0.005$, (b) $(\lambda/\mathcal{R}e) = 0.01$, (c) $(\lambda/\mathcal{R}e) = 0.02$, (d) $(\lambda/\mathcal{R}e) = 0.04$, (e) $(\lambda/\mathcal{R}e) = 0.1$, (f) $(\lambda/\mathcal{R}e) = 1$.

parameter λ/Re upon the components of the secondary flow. For very small values of λ/Re increases the fluid flow will be similar to the non-Newtonian fluid for small values of β, compare Figs. 94*f*) and (5*e*), from which it follow that the contribution of λ/Re tothe secondary flow dominate for small values of β and Reynolds number in case of non-Newtonian fluids.

Fig. (6*a*) shows the pattern of the secondary flow for Newtonian fluid in a curved pipe ($\lambda/Re = 0$) which consist of two symmetric vortexes and as β increases (non-Newtonian), it is notices there is shifting in the centres of the vortexes toward the inner bend of the cross-section of the pipe, Figs. (6*a*, 6*c*).

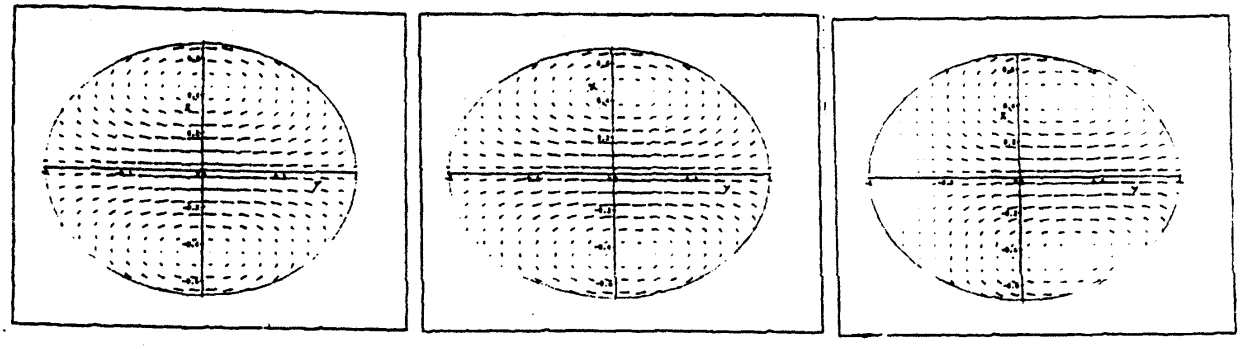

FIG. 6. Secondary flow components for $c = 0.65$, $(\lambda/\mathcal{R}e) = 0$. (a) $\beta = 0$. (b) $\beta = 0.1$, (c) $\beta = 0.3$.

Finally, from eq. (27), it is easy to prove that as $\lambda/Re \to \infty$ the secondary flow in a straight pipe is given by

$$v_x = U\,\tau a\,(1 - x^2 - c^2 y^2)\,(8c^2 h - 1)\,y$$

and

$$v_y = U\,\tau a\,(1 - x^2 - c^2 y^2)\,(-8h + 1)\,x.$$

Using the definition of streamline, thelast equation may write as

$$\frac{v_x}{v_y} = \frac{dx}{dy} = \frac{(8c^2 h - 1)\,y}{(-8h + 1)\,x}$$

On solving this ordinary differential equation, we obtain the streamlines curves as

$$y^2 - \frac{(-8h + 1)}{(8c^2 h - 1)}\,x^2 = const. \qquad \ldots (28)$$

which shows the secondary flow streamline patterns are hyperbolae and the feature of this streamline for an elliptical cross-section are given in Fig. 7.

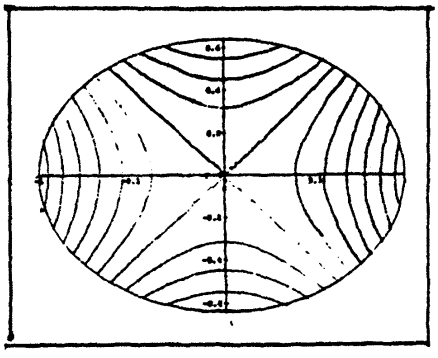

FIG. 7. Secondary flow components in straight pipe

6.2. AXIAL VELOCITY

The surface u = constant have been plotted for different values for the parameters λ/Re and β. Fig. (8) shows the effect of the parameter λ/Re on the helical flow for non-Newtonian fluid, β has been

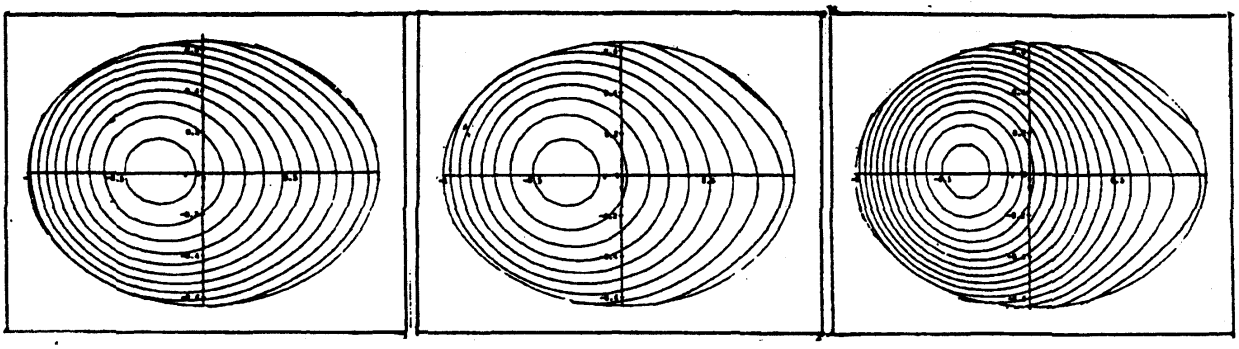

FIG. 8. Axial velocity fo β = 0.1. (a) $(\lambda/\mathcal{R}e)$ = 0.005, (b) $(\lambda/\mathcal{R}e)$ = 0.004, (c) $(\lambda/\mathcal{R}e)$ = 0.0033

set at 0.1. It is observed that for small values of λ/Re (Re varied from 40 to 60), the secondary motion push the maximum axial velocity away from the center of the cross-section toward the outside of the bend. This implies that the centrifugal force, which generates the secondary motion, is not symmetric inside the cross-section. Accordingly, the secondary motion itself is not symmetric inside the cross-section, as we have seen in the previous section.

Fis. (8*a*) and (9) show the effect of β upon the axial flow, it is noticed that as β increases from 0.01 to 1, the curvature of the streamline increases and the motion become more thicked near the outside of the bend.

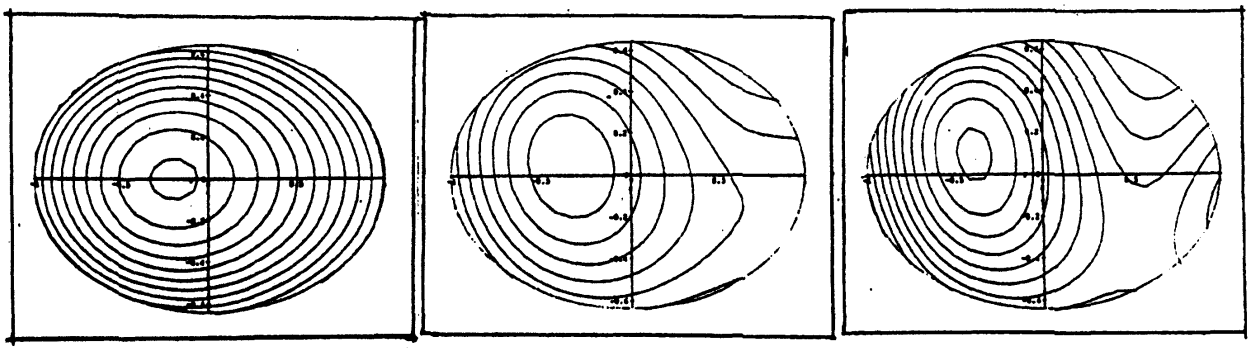

FIG. 9. Axial velocity for $(\lambda/\mathcal{R}e)$ = 0.005. (a) β = 0.01, (b) β = 0.5, (c) β =1

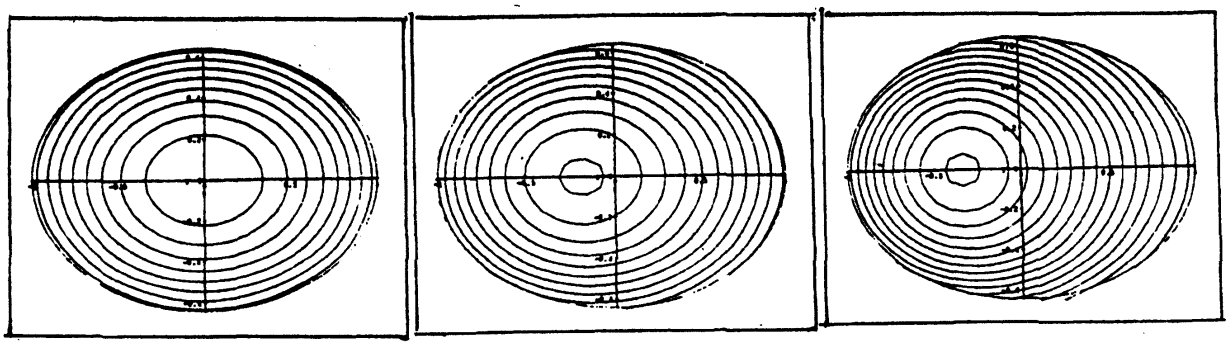

FIG. 10. Axial velocity for β = 0. (a) $(\lambda/\mathcal{R}e)$ = 0, (b) $(\lambda/\mathcal{R}e)$ = 0.005, (c) $(\lambda/\mathcal{R}e)$ = 0.0033.

Finally, Figs. (10*b*) and (10*c*) shows the effect of λ/Re on the axial flow for Newtonian fluid flow in a helical pipe with an elliptical cross-section, while Fig. (10*a*) illustrate the axial velocity in case of curved pipe.

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