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AN ALGORITHM FOR BILEVEL FRACTIONAL PROGRAM WHEN THE FOLLOWER CONTROLS FEW VARIABLES

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This paper presents an algorithm for solving a bilevel programming problem in which both the decision makers wish to maximize their own linear fractional objective function. The algorithm finds an optimum of a Bilevel Fractional Programming (BFP) problem if one exists. The algorithm works efficiently if the lower decision maker does not control many variables. An example to illustrate the algorithm is given at the end.

Key Words : Bilevel Linear Fractional Programming

1. INTRODUCTION

Bilevel Programming (BLP) is developed to solve decentralized planning problems in which decisions are made in a hierarchical order. Each Decision Maker (DM) independently controls a subset of the decision variables and the DM have different perhaps conflicting objective functions. The upper level DM is called leader and the lower level DM is called follower. The follower executes its policies after and in view of the leader, who in turn optimizes its objective independently over the reaction of the follower.

Such problems occur in hierarchical administrative structures where the DM at different levels of seniority sharing a common pool of resources control different set of variables[2-5]. Some of the other applications are in government departments, setting of penalties for illegal drug import, fixing of import quotas, development of transportation and communication infrastructure, spatial competition facility location, signal optimization and traffic assignment.

This paper gives an algorithm to solve a Bilevel Fraction Program (BFP) given by

$$\max_{x} \; F(x, y)$$

where y solves

$$\max_{y} \; f(x, y)$$

subject to $\quad Ax + By \leq t,$

where $F(x, y)$ and $f(x, y)$ are linear fractionals, $X \in R^{n_1}, y \in R^k, t \in R^m$. A and B are given $m \times n_1$ and $m \times k$ matrices respectively. It is assumed that $m > k$, and rank $(B) = k$.

2. Mathematical Formulation

Let $S = \{(x, y) \mid Ax + By \leq t\}$

The Linear Fractional Bilevel Programming Problem is[7] :

$$\underset{x}{\text{Max}}\ F(x, y) = \frac{ax + by + \alpha}{cx + dy + \beta},$$

where y solves

$$\underset{y}{\text{Max}}\ f(x, y) = \frac{rx + py + \gamma_1}{sx + qy + \delta_1}$$

subject to $(x, y) \in S$,

where $a^t, c^t, r^t, s^t \in R^{n_1}$, $b^t, d^t, p^t, q^t \in R^k$, where superscript "$t$" denotes transpose.

It is assumed that $cx + dy + \beta > 0$ and $sx + qy + \delta_1 > 0$ for all (x, y) in S.

Because x is fixed prior to the maximization of f, the above problem becomes

$$\text{(BFP')}\ \underset{x}{\text{Max}}\ F(x, y) = \frac{ax + by + \alpha}{cx + dy + \beta},$$

where y solves

$$\underset{y}{\text{Max}}\ f_1(y) = \frac{py + \gamma}{qy + \delta}$$

subject to $By \leq t - Ax$,

where $\gamma = rx + \gamma_1$, $\delta = sx + \delta_1$ for fixed x.

For a given $x = \bar{x}$, let $Q(\bar{x}) = \{y : By \leq t - A\bar{x}\}$.

Then the set of optimal solutions to the follower's problem is given by

$$Y(\bar{x}) = \{y^* \mid f_1(y^*) \geq f_1(y)\ \forall\ y \in Q(\bar{x})\}.$$

The set of rational reactions of the follower over S or the leader's solution space is then represented by

$$S' = \{(x, y) \mid (x, y) \in S, y \in Y(x)\}$$

Definition 1 — A solution (x, y) in S is called a feasible solution if $(x, y) \in S'$.

Definition 2 — A feasible solution (x^*, y^*) is an optimal solution if

$$F(x^*, y^*) \geq F(x, y) \ \forall \ (x, y) \in S'.$$

We assume S and S' are bounded and nonempty. This guarantees the existence of optimal solution.

Let $\boldsymbol{B}_i$ be a nonsingular submatrix consisting of exactly k distinct rows of $\boldsymbol{B}$.

Now consider the problem,

$$(FP_i) \ : \ \text{Max } f_1(y) = \frac{py + \gamma}{qy + \delta}$$

subject to $\quad \boldsymbol{B}_i \, y \leq v$

Then the following lemma holds due to nonsingularity of the matrix $\boldsymbol{B}_i$.

Lemma 1 (*a*) — If (FP_i) has a unique optimal solution for any v, then it has a unique optimal solution for all vectors v.

(*b*) If the optimal value of (FP_i) is bounded for some v, it is bounded for all v.

Let A_i be the submatrix of A which consists of the rows that correspond to the rows of $\boldsymbol{B}$ which were selected to form the non-singular matrix $\boldsymbol{B}_i$. Let

$$A = \begin{pmatrix} A_i \\ \overline{A}_i \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} B_i \\ \overline{B}_i \end{pmatrix} t = \begin{pmatrix} t_i \\ t_i \end{pmatrix}$$

Then the set of constraints of (BFP) become

$$\boldsymbol{A}_i x + \boldsymbol{B}_i y \leq t_i, \qquad \text{... (1)}$$

and

$$\overline{\boldsymbol{A}}_i x + \overline{\boldsymbol{B}}_i y \leq \overline{t}_i \qquad \text{... (2)}$$

and we have the following lemma :

Lemma 2 — If (FP_i) is bounded then (x^*, y^*) is an optimal solution of (FP_i) where

$$y^* = \boldsymbol{B}_i^{-1} (t_i - A_i x^*)$$

When $\quad (\overline{x}, \overline{y}) = (\overline{x}, \boldsymbol{B}_i^{-1} (t_i - \boldsymbol{A}_i \overline{x}))$, then

$$\overline{y} = \boldsymbol{B}_c^{-1} (t_i - A_i \overline{x})$$

$$\Rightarrow \boldsymbol{B}_i \overline{y} = t_i - \boldsymbol{A}_i x$$

$$\Rightarrow \boldsymbol{A}_i \overline{x} + \boldsymbol{B}_i \overline{y} = t_i$$

i.e., equality holds in (1). If $(\overline{x}, \overline{y})$ also satisfies (2), then by the following Theorem 1, we have $(\overline{x}, \overline{y})$ is a feasible solution of (BFP').

***Theorem* 1** — *Suppose that* (FP_i) *has an optimal solution. Then* $(\bar{x}, \bar{y}) \in S$ *implies* $(\bar{x}, \bar{y}) \in S'$ *where* $\bar{y} = B_i^{-1}(t_i - A_i\bar{x})$

PROOF : Suppose $(\bar{x}, \bar{y}) \in S \nRightarrow (\bar{x}, \bar{y}) \in S'$,
i.e., for a given $x = \bar{x}$, let $y = y'$ be an optimal solution of the follower's problem

$$\max_y \; f_1(y) = \frac{py + \gamma}{qy + \delta}$$

subject to $A\bar{x} + By \le t$. ... (3)

Since $(\bar{x}, \bar{y}) \in S, \bar{y}$ is feasible to (3). Since $\bar{y}$ is not an optimal solution, therefore,

$$f_1(y') > f_1(\bar{y}).$$

Consider the following linear fractional programming problem

$$(FP_i) \;\; \text{Max} \; f_1(y) = \frac{py + \gamma}{qy + \delta}$$

subject to $B_i y \le v$,

where $v = t_i - A_i\bar{x}$.

By assumption and Lemma 1, this (FP_i) has a solution and by Lemma 2 a solution is $y = B_i^{-1}(t_i - A_i\bar{x}) = \bar{y}$.

Since y' is feasible to (3), it is feasible for (FP_i) and $f_1(y') > f_1(\bar{y})$ contradicting that $\bar{y}$ is an optimal solution for (FP_i). Therefore, $y' = \bar{y}$ and $\bar{y}$ is an optimal solution for (3). Therefore, $(\bar{x}, \bar{y}) \in S'$.

Thus we have got feasible solutions for the given problem (BFP). Next theorem gives us the optimal solution to the given problem.

***Theorem* 2** — *Suppose that* $p^* = (x^*, y^*)$ *is an optimal basic solution of the problem* (BFP). *Then there exists k constraints such as that when* A_i *and* B_i *are formed from these k rows of* A *and* B *respectively,* B_i *is nonsingular,* $y^* = B_i^{-1}(t_i - A_i x^*)$, *so*

$$p^* = (x^*, B_i^{-1}(t_i - A_i x^*)) \text{ and the problem}$$

$$\max_y \; f_1(y) \text{ subject to } B_i y \le t_i - A_i x^* \text{ is bounded.}$$

PROOF : Consider the following linear fractional programming problem

$$(FP) : \text{Max} \; f_1(y) = \frac{py + \gamma}{qy + \delta}$$

subject to $By \le t - Ax^*$.

Applying Charnes and Cooper technique[5] to linearize (*FP*) we get the following linear programming problem

$$(LP) : \text{Max } pz + rl$$

sublect to $$Bz - (t - Ax^*)\,1 \leq 0$$

$$qz + \delta l = 1$$

$$l > 0$$

where $$z = ly.$$

Let (z^*, l^*) be the optimal solution of (LP). Then $l^* > 0$ and $y^* = \frac{z^*}{l^*}$ will be the optimal solution of (FP). Since rank of $B = k$ it will have k non- zero components. Also then

$$B_i z^* - (t_i - A_i x^*)\, l^* = 0$$

and $$qz^* + \delta l^* = 1, \quad z^* \geq 0, l^* > 0$$

$$\Rightarrow \frac{B_i z^*}{l^*} = t_i - A_i x^*$$

$$B_i y^* = t_i - A_i x^*$$

$$y^* = B_i^{-1} (t_i - A_i x^*)$$

and $$q\frac{z^*}{l^*} + \delta = \frac{1}{l^*} \Rightarrow qy^* + \delta = \frac{1}{l^*}$$

Objective function value in (LP)

$$= p \cdot z^* + \gamma l^*$$

$$= l^* (py^* + \gamma)$$

$$= \frac{py^* + \gamma}{qy^* + \delta}$$

= objective function value in (*FP*).

Obviously $(x^*, B_i^{-1} (t_i - A_i x^*)) = (x^*, y^*)$ is optimal solution of the given problem.

Based on the above discussion we have the following algorithm to solve the given problem (*BFP*).

3. ALGORITHM

Step 1 — For each set i of k constraints form the matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ and check $\boldsymbol{B}_i$'s for nonsingularity. If $\boldsymbol{B}_i$ is singular discard it otherwise go to Step 2.

Step 2 — Check if the problem

$$(\boldsymbol{FPB}_i) : \text{Max } f_1(y)$$

subject to $\quad \boldsymbol{B}_i y \leq 0$

is bounded. If unbounded discard $\boldsymbol{B}_i$, otherwise solve it. Let y^* be its optimal solution. Go to Step 3.

Step 3 — Solve the leader's problem

$$(\boldsymbol{FPO}_i) \text{ — Max } \frac{ax + by^* + \alpha}{cx + dy^* + \beta}$$

subject to $\quad \boldsymbol{Ax} + \boldsymbol{By}^* \leq t,$

where $\quad y^* = \boldsymbol{B}_i^{-1}\ (t_i - \boldsymbol{A}_i x)$

Step 4 — The solution obtained in Step 3 with the maximum value of $F(x, y)$ is the optimal solution of the problem (BFP).

Example — $\displaystyle \operatorname*{Max}_{x} \frac{x}{y_1 + y_2 + 1}$

where $\quad y_1, y_2$ solve

$$\operatorname*{Max}_{y_1, y_2} \frac{y_1}{y_2 + 2}$$

subject to
$$x + y_1 + y_2 \leq 10$$
$$x \leq 8$$
$$y_1 \leq 9$$
$$y_2 \leq 7$$
$$-x \leq 0$$
$$-y_1 \leq 0$$
$$-y_2 \leq 0$$

The set of constraints is

$Ax + By \le t$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$t = \begin{pmatrix} 10 \\ 8 \\ 9 \\ 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Rank of B = 2.

The matrices which are nonsingular are :

$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -B_3 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, B_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B_7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B_8 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(FBP_1) : \text{Max } \frac{y_1}{y_2 + 2}$$

subject to $y_1 + y_2 \le 0$

$y_1 \le 0$

This problem is not regular. It is not defined at $y_2 = -2$. Similarly problems (FPB_2), (FPB_3), (FPB_5) and (FPB_7) are not regular. So we discard matrices B_1, B_2, B_3, B_5 and B_7.

$$(FPB_8) : \text{Max } \frac{y_1}{y_2 + 2}$$

subject to $-y_1 \le 0$

$-y_2 \le 0.$

This problem is unbounded. Therefore, B_8 is also discarded.

Consider the problem

$$(FPB_4): \text{ Max } \frac{y_1}{y_2+2}$$

subject to $\quad y_1 + y_2 \le 0$

$$-y_2 \le 0.$$

This problem is bounded with optimal objective value zero and solution

$$B_4^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \text{ and } y_4^* = B_4^{-1}(t_4 - A_4 X) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 10-x \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 10-x \\ 0 \end{pmatrix} \; y^* = \begin{pmatrix} 10-x \\ 0 \end{pmatrix}$$

Solve the problem

$$(FPO_4) \text{ Max } \frac{x}{10-x+1}$$

subject to

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 10-x \\ 0 \end{pmatrix} \le \begin{pmatrix} 10 \\ 8 \\ 9 \\ 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow x \le 8,\; -x \le -1 \text{ and } x \ge 0$$

It's optimal solution is $x^* = 8$ and optimal objective value is $\frac{8}{3}$.

$$y^* = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 10-8 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$\therefore$ The solution of (BFP) is $(x^*, y_1^*, y_2^*) = (8, 2, 0)$. The problem (FPB_6) also is bounded with optimal objective value equal to zero and optimal solution of the problem (*BFP*) is (1, 9, 0) with objective value $\frac{1}{10}$. Hence the best solution of (*BFP*) is (8, 2, 0).

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REFERENCES

1. J. F. Bard, *Naval Res. Logistic Q.* **33** (1984) 13-26.
2. J. F. Bard, *O R* **31** (1983) 670-84.
3. J. F. Bard, *O R.* **8** (2) 260-72.
4. W. F. Bialas and M. H. Karwan, *Mgmt. Sci.* **30** 1984) 8, 1004-20.
5. W. Candler and R. J. Townsley, *Comput. O R.* **9** (1982) 59-76.
6. A. Charnes and W. W. Cooper, *Naval Res. Logistic. Q.* **9** (1962) 181-86.
7. D. Thirwani and S. R. Arora, *Bilevel Linear Fractional Programming Probem*, Cahiers due Cero, Belgium, **35** (1-2) (1993) 135-49.