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NORM OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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In this paper we give sufficient conditions to characterise the set of essentially bounded measurable functions ϕ on the open unit disk D for which $\| T_\phi \| = \| \phi \|_\infty$ where T_ϕ is the Toeplitz operator with symbol ϕ on the Bergman space $L_a^2(D)$.

Key Words : Toeplitz Operator; Bergman Space Hilbert Space; Hankel Operator; Berezin Transform

Let $D = \{z \in C : |z| < 1\}$ be the open unit disc in the complex plane C. Let $dA(z)$ be the area measure on D normalised so that the area of the disc D is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dx\, dy = \frac{1}{\pi} r dr\, d\theta.$$

Let $L^2(D, dA)$ denote the Hilbert space of complex-valued, absolutely square-integrable, Lebesgue measurable functions f on D with the inner product

$$\langle f, g \rangle = \int f(z)\, \overline{g(z)}\, dA(z).$$

Let $L^\infty(D, dA)$ denote the Banach space of Lebesgue measurable functions f on D with $\| f \|_n = ess\ \sup \{|f(z)| : Z \in D\} < \infty$. Let $H^\infty_{(D)}$ be the space of bounded analytic functions on D. Let $L_a^2(D)$ (the subscript "a" stands for analytic) be the subspace of $L^2(D, dA)$ consisting of analytic functions. The space $L_a^2(D)$ is called the Bergman space. Since point evaluation at $z \in D$ is a bounded linear functional on the Hilbert space $L_a^2(D)$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(D)$ such that

$$f(z) = \int_D f(w)\, \overline{K_z(w)}\, dA(w)$$

for all f in $L_a^2(D)$. Let $K(z, w)$ be the function on $D \times D$ defined by

$$K(z, w) = \overline{K_z(w)}.$$

The function $K(z, w)$ is thus the reproducing kernel for the Bergman space $L_a^2(D)$ and is called the Bergman kernel. It can be shown that the sequence of functions $\{e_n(z)\} = \left\{\sqrt{n+1}\, z^n\right\}_{n\geq 0}$ form the standard orthonormal basis for $L_a^2(D)$ and $K(z, w) = \sum_{n=1}^{\infty} e_n(z)\,\overline{e_n(w)}$. The Bergman kernel is independent of the choice of orthonormal basis and $K(z, w) = \frac{1}{(1-z\bar{w})^2}$. Since $L_a^2(D)$ is a closed subspace of $L^2(D, dA)$ (see [15]); there exists an orthogonal projection P from $L^2(D, dA)$ onto $L_a^2(D)$. For $\phi \in L^\infty(D)$, we define the Toeplitz operator T_ϕ on $L_a^2(D)$ by $T_\phi f = P(\phi f), f \in L_a^2(D)$. The big Hankel operator H_ϕ is mapping from $L_a^2(D)$ into $(\overline{L_a^2(D)})^\perp$ defined by $H_\phi f = (I-P)(\phi f), f \in L_a^2(D)$. The little Hankel operator h_ϕ is a mapping from $L_a^2(D)$ into $\overline{L_a^2(D)}$ defined by $h_\phi f = \bar{P}(\phi, f), f \in L_a^2(D)$ where $\bar{P}$ is the projection from $L^2(D, dA)$ onto $\overline{L_a^2(D)}$. There are also many equivalent ways of defining little Hankel operators. For example, define the map S_ϕ from $L_a^2(D)$ into $L_a^2(D)$ by $S_\phi f = P(J(\phi f))$ where J is the mapping from $L^2(D)$ into $L^2(D)$ defined by $J(h(z)) = h(\bar{z})$. It is easy to see that the operator S_ϕ is unitarily equivalent to the little Hankel operator h_ψ for some $\psi \in L^\infty(D)$.

Let T denote the unit circle in C. Let $L_{C^n}^2(T)$ Denote the Hilbert space of C^n-valued, norm-square integrable, measurable functions on T and $\mathcal{H}_{C^n}^2(T)$ the corresponding Hardy space of functions in $L_{C^n}^2(T)$ with vanishing negative Fourier coefficients. If Φ is a bounded, measurable $M_n = M_n(C)$-valued function (the algebra of $n \times n$ matrices with complex entries) in $L_{M_n}^\infty(T)\,(= L^\infty \otimes M_n)$, then T_Φ denotes the Toeplitz operator defined on $\mathcal{H}_{C^n}^2(T)$ by $T_\Phi f = P(\Phi f)$ where P is the orthogonal projection of $L_{C^n}^2(T)$ onto $\mathcal{H}_{C^n}^2(T)$. A function $\Theta \in H_{M_n}^\infty\,(= H^\infty \otimes M_n)$ is said to be inner if $\Theta(z)$ is an isometry for almost all $z \in T$. It is not so difficult to show in the case of $\mathcal{H}_{C^n}^2(T)$ that the norm of a Toeplitz operator is equal to the norm of its symbol. We present a new proof of the fact to insist that it so happens because the Douglas-Rudin theorem[7] holds in the space $L_{M_n}^\infty$ and because the inner matrix is an isometry.

Theorem 1 — *If* $\Phi \in L^{\infty}_{M_n}(T)$ *then* $\| T_{\Phi} \| = \| \Phi \|_{\infty}$.

PROOF : Let $\mathcal{N} = \{\Theta^* \Psi : \Theta \in H^{\infty}_{M_n}$ is inner and $\Psi \in H^{\infty}_{M_n}(T)\}$. By [7], $\mathcal{N}$ is norm-dense in $L^{\infty}_{M_n}(T)$. Now let $\Phi \in \mathcal{N}$ and $\Phi = \Theta^* \Psi$ where Θ is an inner function in $H^{\infty}_{M^n}(T)$ and $\Psi \in H^{\infty}_{M_n}(T)$. Let $q = \det \theta$. Then it is easy to check that q is an inner function. $H^{\infty}(T)$ and $\Theta^* q \in H^{\infty}_{M_n}(T)$. Thus $\| T_{\Phi} \| \geq \sup_{f \in \mathcal{H}^2_{C^n}(T), \| f \| = 1} \| T_{\Phi} f \| = \| \Phi_q \|_{\infty}$. Since $T_{\Phi} = P M_{\Phi}$ on $\mathcal{H}^2_{C^n}(T)$, $\| P \| = 1$ and $\| M_{\Phi} \| = \| \Phi \|_{\infty}$ (cf. [14]) it follows that $\| T_{\Phi} \| \leq \| \Phi \|_{\infty}$. Therefore, $\| T_{\Phi} \| = \| \Phi \|_{\infty}$. Now since $\mathcal{N}$ is dense in $L^{\infty}_{M_n}$, the result follows. ■

This idea can also be applied to the case of a totally ordered group as follows.

Let $(G, \leq)$ be an ordered group (i.e., $(G, \leq)$ is a pair consisting of an abelian (discrete) group G and a partial order $\leq$ on G such that for all $x, y, z \in G$ the implication $x \leq y \Rightarrow x + z \leq y + z$ holds, and either $x \leq y$ or $y \leq x$.), $\hat{G}$ be the Pontryagin dual group of G and set $G^+ = \{x \in G : 0 \leq x\}$. The set $\hat{G}$ is compact[12]. If $f \in L^1(\hat{G})$, denote by $\hat{f} : G \to C$ its Fourier transform,

$$\hat{f}(x) = \int_{\hat{G}} f(\gamma) \overline{\gamma(x)} \, dm(\gamma), \; (x \in G).$$

Here m is the unique Haar measure on $\hat{G}$ such that $m(\hat{G}) = 1$, and $L^p(\hat{G}) = L^p(\hat{G}, m)$. The generalised Hardy space $H^p = H^p(G, \leq)$ is defined to be

$$H^p = \{f \in L^p(\hat{G}) : \hat{f}(x) = 0 \; (x \in G, x \leq 0)\}.$$

This is an L^p-closed vector subspace of $L^p(\hat{G})$ for all $p \in [1, \infty]$. Let $L^{\infty}(\hat{G}, m)$ be the class of essentially bounded, complex-valued, measurable functions on $\hat{G}$ with the essential supremum norm. A function $\Theta \in H^2(G)$ is said to be an inner function [3] if $|\Theta| = 1$ (almost everywhere on G). Denote by P_G the projection of the Hilbert space $L^2(\hat{G})$ onto H^2. If $\phi \in L^{\infty}(\hat{G})$, define $T_{\phi} \in \mathcal{L}(H^2)$ by setting $T_{\phi} f = P_G(\phi f)$, and call T_{ϕ} a (generalised) Toeplitz operator on H^2. The following theorem follows from a result of Bernard, Garnett, Marshall[3] which generalises the Douglas-Rudin theorem to the case of (totally) ordered group.

Theorem 2 — *If G is a totally ordered group, then the set $\{\overline{\Theta}\Psi, \Theta, \Psi \in H^\infty(G), \Theta$ inner$\}$ is dense in $L^\infty(\hat{G})$.*

PROOF : Let $L^\infty \to \hat{L^\infty}$, $\Phi \mapsto \hat{\Phi}$, be the Gelfand representation and Ω be the character space of L^∞, so $\hat{L^\infty} = C(\Omega)$. Since H^∞ is strongly log modular (this means every $\Theta = \overline{\Theta}$ in L^∞ is of the form $\Theta = \log |f|$ where f is an invertible element of H^∞), a result of Bernard, Garnett and Marshall[3] implies that the set $\{\hat{h} : h \in H^\infty, h$ inner$\}$ separates the points of Ω. By the Stone-Weierstrass theorem[6]. $\hat{L^\infty}$ is the closed linear span of all $\hat{\overline{\Theta}}\hat{\Psi}$ with Θ, Ψ inner functions. ■

The Corollary below was also observed by Murphy[12]. We stress the fact that this happens since the inner function is an isometry on H^2 and because the Douglas-Rudin theorem holds for $L^\infty(\hat{G})$ if G is a totally ordered group.

Corollary 3 — If G is a totally ordered group and $\phi \in L^\infty(\hat{G})$ then $\| T_\phi \| = \| \phi \|_\infty$.

PROOF : The proof is similar to Theorem 1. ■

Now in the case of the Bergman space there exists $\phi \in L^\infty(D)$ such that the norm of the Toeplitz operator is not equal to the norm of its symbol. For example, consider the function $\phi(z) = \exp\left(i \ln \ln \frac{1}{|z|^2} \right)$. Note ϕ is unimodular on D and T_ϕ = diag $(\overline{e^{iln(n+1)}}\, \Gamma(i+1))$. Thus $T_\phi^* T_\phi$ = diag $(|\Gamma(i+1)|^2)$ and $\| T_\phi \|^2 \leq |\Gamma(i+1)|^2 = \frac{\pi i}{\sin \pi i} < 1 = \| \phi \|_\infty$. So the question that arises here is to find out the largest subclass $\mathcal{P}$ of $L^\infty(D)$ such that the norm of a Toeplitz operator with symbol in $\mathcal{P}$ is equal to the norm of its symbol. We clearly need to increase our insight into this phenomenon.

The following theorem establishes that on the Bergman space $L_a^2(D)$ the Toeplitz operator T_ϕ represents a multiplication operator on a closed subspace of $L_a^2(D)$ if and only if the symbol ϕ is analytic.

Theorem 4 — *Let S be a subspace of $L^\infty(D)$ such that for $\phi \in S$ there exists a closed subspace $\mathcal{M} \neq \{0\}$ of $L_a^2(D)$ for which $T_\phi f = \phi f$, for all $f \in \mathcal{M}$. Then $S \subset H^\infty(D)$.*

PROOF : Suppose $T_\phi f = \phi f$, for some fix $f \in \mathcal{M} \subset L_a^2$. Therefore, $\phi(z) = \frac{T_\phi f(z)}{f(z)}$. Hence ϕ is analytic on D - {zeros of f}. Each isolated singularity of ϕ in D is removable since ϕ is assumed to be bounded. Thus ϕ is analytic on D. Since $\phi \in L^\infty$ hence, $\phi \in H^\infty(D)$. ■

Thus we have shown that the kernel of a big Hankel operator on the Bergman space is trivial in contrast to the Hardy space case where the kernel of a Hankel operator is an invariant subspace of the unilateral shift. But it is important to note that the kernel of a little Hankel operator is an invariant subspace of the Bergman shift operator and that the Toeplitz operator is related to the little Hankel operator in some asymptotic sense[13]. To study the Bergman space case in detail we need to introduce the following notation.

Let *Aut (D)* denote the set of analytic, bijective maps of D onto D. A function ϕ on D is in *Aut (D)* if and only if there is a $z \in D$ such that $\phi(w) = e^{i\theta}\frac{z-w}{1-\bar{z}w}$ for $w \in D$ and θ real. For z, w in D, let $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$. Recall that a continuous function on the disc D is harmonic if and only if it has the mean value property. A function $u \in C(D)$ is said to have the invariant mean value property if

$$\int_0^{2\pi} u(\phi_z(re^{i\theta}))\frac{d\theta}{2\pi} = u(\phi_z(0))$$

for every $\phi_z \in Aut(D)$ and every $r \in [0, 1)$. Here "invariant" means conformal invariance, i.e., invariant under composition with elements of *Aut (D)*. If u is harmonic on D, then so is $u \circ \phi_z$ for every $\phi_z \in Aut(D)$. Hence, harmonic functions have the invariant mean value property. Conversely, if a function $u \in C(D)$ has the invariant mean value property, then u is harmonic on D[1]. The invariant mean value property concerns averages over circles with respect to arc length measure. In the Bergman space L_a^2, it is natural to define an invariant condition stated in terms of an area average over D. A function $u \in C(D) \cap L^1(D, dA)$ has the area version of the invariant mean value property if $\int_D u \circ \phi_z \, dA = u(\phi_z(0))$ for every $\phi_z \in Aut(D)$. If u is in $C(D) \cap L^1(D, dA)$, then so is $u \circ \phi_z$ for every $\phi_z \in Aut(D)$, so the left-hand side of the above equation makes sense. If u is harmonic on D and in $L^1(D, dA)$, then so is $u \circ \phi_z$ for every $\phi_z \in Aut(D)$. Thus, by the mean value property, harmonic functions have the invariant area mean value property. If $u \in C(\overline{D})$ satisfies the area version of the invariant mean value property then u is harmonic[1]. If $u \in L^\infty$ and has the area version of the invariant mean value property then also u is harmonic[8]. The following result is a consequence of it.

Theorem 5 — *If ϕ is a bounded harmonic function on D, then* $\| T_\phi \| = \| \phi \|_\infty$.

PROOF : Let $\tilde{\phi}(z) = \langle T_\phi k_z, k_z \rangle$, where k_z is the normalised reproducing kernel for the Hilbert space $L_a^2(D)$. Hence, $\tilde{\phi}(z) = \int_D \phi(w) |k_z(w)|^2 \, dA(w), z \in D$. Since $\phi_z' = -k_z$, making a change of variable we also have $\tilde{\phi}(z) = \int_D \phi(\phi_z(w)) \, dA(w) = \phi(\phi_z(0)) = \phi(z), z \in D$. Thus $|\phi(z)| = |\tilde{\phi}(z)| = |\langle T_\phi k_z, k_z \rangle| \le \| T_\phi \|$ for all $z \in D$. Hence $\| \phi \|_\infty \le \| T_\phi \|$. Now since the Bergman projection has norm 1, we have $\| T_\phi \| = \| \phi \|_\infty$. ■

Corollary 6 — If ϕ is a real-valued bounded subharmonic function then $\| T_\phi \| = \| \phi \|_\infty$.

PROOF : Suppose $\phi \in L^\infty$ is real-valued. Then the subharmonicity of ϕ implies that

$$\tilde{\phi}(z) = \int_D \phi(\phi_z(w))\, dA(w) \geq \phi(\phi_z(0)) = \phi(z).$$

Hence, $|\tilde{\phi}(z)| \geq |\phi(z)|$ for all $z \in D$. Therefore, $\| \phi \|_\infty \leq \| \tilde{\phi} \|_\infty \leq \| T_\phi \|$. The result follows. ■

Thus we see that only for bounded harmonic functions (subharmonic functions) ϕ in $L^\infty(D)$, the area-version of the mean-value (sub mean-value) property holds and hence $\tilde{\phi}(z) = \phi(z)$ and therefore $\| T_\phi \| = \| \phi \|_\infty$. But there exists functions ϕ in $L^\infty(D)$ that are not subharmonic on the disc D but yet $\| T_\phi \| = \| \phi \|_\infty$ as the following example shows. Again the set of ϕ in $L^\infty(D)$ such that $\| T_\phi \| = \| \phi \|_\infty$ is not a subspace.

Example 7 — Let ϕ be the characteristic function of the annulus $R \leq |z|^2 < 1$. For $i, j \geq 0$, $\langle T_\phi e_j, e_i \rangle = \langle \phi e_j, e_i \rangle = 0$ if $i \neq j$ and $\{ e_n \}_{n \geq 0}$ is the standard orthonormal basis for $L_a^2(D)$. Moreover $\langle T_\phi e_j, e_j \rangle = (j+1) \int_R^1 r^j\, dr = 1 - R^{j+1}$. Hence, the matrix of T_ϕ with respect to the standard orthonormal basis $\{ e_n \}_{n \geq 0}$ is diagonal, $\| T_\phi \| = 1$ and $\| \phi \|_\infty = 1$, but ϕ is not harmonic. It will be difficult to describe those ϕ for which $\| T_\phi \| = \| \phi \|_\infty$, because the above example shows that this set is not even linear (i.e., it is not a subspace): the function 1 (constant 1) and the above ϕ both belong to this set, but their difference does not, because $T_{1-\phi}$ is a diagonal operator with weights R^{j+1}, so $\| 1 - \phi \|_\infty = 1 > R = \| T_{1-\phi} \|$.

One can also define a Toeplitz operator with symbol in $L^2(D, dA)$. In fact, if $\phi \in L^2(D, dA)$ and $\phi \geq 0$ then T_ϕ is bounded if and only if $\tilde{\phi} \in L^\infty(D)$[15]. Thus there exists ϕ that is not essentially bounded on D yet it gives rise to a bounded Toeplitz operator T_ϕ. For example, consider the function $\phi(z) = \log \frac{1}{1 - |z|^2}$. Clearly $\phi \in L^2(D) \setminus L^\infty(D)$, as $\int_D \left| \log \frac{1}{1-|z|^2} \right|^2 dA(z) =$ $\int_0^1 |\log(1-r^2)|^2\, 2r\, dr = \int_0^1 |\log(1-t)|^2\, dt = \int_0^1 |\log t|^2\, dt = \int_0^\infty x^2 e^{-x}\, dx = \Gamma(3) < \infty$. But we can also

show that $\tilde{\phi} \in L^{\infty}(D)$. In view of this it is important to know the set of functions in $L^1(D, dA)$ for which mean value property holds. Axler and Cuckovic[1] have showed that if the constant functions are the only functions $V \in C([0, 1)) \cap L^1[0, 1]$ such that

$$(*)\quad V(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} V(s)ds$$

for every $t \in (0, 1)$ then functions $u \in C(D) \cap L^1(D, dA)$ having the area version of the invariant mean value property is harmonic. Thus we want to know the solutions of the integral operator equation (*). Casting it in a slightly different form, we want to show that the eigenspace corresponding to the eigenvalue 1 of the operator T is one dimensional, where $(Tu)(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} u(s)\, ds$ defined on $C([0, 1)) \cap L^1[0, 1]$. The operator T is an integral operator with positive kernel which maps $L^1[0, 1]$ to $C([0, 1))$, so the domain may as well be considered as $L^1[0, 1]$. Again T does not leave L^1 invariant because $G(s) = \int_0^1 (1-t)^2 \frac{1+ts}{(1-ts)^3} dt$ is an unbounded function of s and hence there exist $0 \le f \in L^1$ such that $\int_0^1 f(s)\, G(s) ds = \infty$. We could regard T as an operator on $C([0, 1))$, which is an invariant subspace of T (because $(1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} ds = 1$, for details see Theorem 8). The difficulty here is that there is no natural norm on $C([0, 1))$, although there is a metric which makes $C([0, 1))$ into a Frechet space. Again we know in D, the only measure left invariant by all Mobius transformations is the pseudo-hyperbolic measure

$$d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

Therefore, the only harmonic function in $L^p(D, d\eta)$ is constant zero. To see this e.g. for $L^2(D, d\eta)$, denote

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt.$$

This is a nonnegative and nondecreasing function of r. Further,

$$\|f\|^2_{L^2(D, d\eta)} = \int_0^1 M(r) \frac{2r}{(1-r^2)^2} dr < \infty.$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f = 0$. Thus even though one can show that every space $L^p\left((0,1), \frac{dt}{(1-t)^2}\right)$, $1 \le p \le \infty$ is an invariant subspace[8] of T but these spaces are no good in this context. This is because (except for L^∞) the corresponding spaces $L^p(D, d\eta)$ do not contain nonzero harmonic functions, even no nonzero constants. We shall show below that T leaves $L^\infty[0,1]$ invariant, is continuous on that space and has spectral radius 1, again on that space. Moreover we shall show that any fixed points of T other than the constants have unbounded variation and do not attain their supremum or infimum on $[0, 1)$.

Theorem 8 — *The operator T is a bounded operator from $L^\infty[0,1]$ into $L^\infty[0,1]$ and has spectral radius 1. Moreover, any fixed points of T other than the constants have unbounded variation and do not attain their supremum or infimum on $[0, 1)$.*

PROOF : The points to note are :

(*a*) Constant functions are fixed points.

(*b*) If $u \ge 0$ on $[0,1)$, $u \in C([0,1))$ and u is not identically zero then $Tu > 0$ on $[0, 1)$.

We shall prove these as follows: For $0 < r < 1$, $k = |\bar{y} r|$ and $R = |y|^2 < 1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-\bar{y} re^{i\theta}|^4} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-ke^{i\theta}|^4} d\theta$$

$$= \frac{1}{2\pi i}\int_T (1-kz)^{-2}\left(1-\frac{k}{z}\right)^{-2}\frac{dz}{z} = \frac{1+Rr^2}{(1-Rr^2)^3}. \text{ Thus}$$

$$(1-t)^2\int_0^1 \frac{1+ts}{(1-ts)^3} ds = (1-t)^2\int_0^1 \frac{1+r^2 t}{(1-r^2 t)^3} 2r\, dr =$$

$$(1-|y|^2)^2\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-\bar{y} re^{i\theta}|^4} d\theta\right) 2r\, dr =$$

$$\int_0^1 (1-|y|^2)^2\left(\sum_{n=0}^{\infty} (n+1)^2 r^{2n}|y|^{2n}\right) 2r\, dr = (1-|y|^2)^2\sum_{n=0}^{\infty}(n+1)|y|^{2n} = 1.$$

Thus it follows that if $u \in L^\infty$ then $Tu \in L^\infty$ and $\|Tu\|_\infty \le \|u\|_\infty$. Therefore, $\|T\| \le 1$. When u is a constant function, we have $\|Tu\|_\infty = \|u\|_\infty$. Hence, $\|T\| = 1$ and the spectral radius of T is 1. From these, it also follows that if u is a nonconstant fixed point in $C([0, 1))$ then

1. $\inf_{[0,1)} u < u(t)$ for all $t \in [0, 1)$.

2. $\sup_{[0,1)} u > u(t)$ for all $t \in [0, 1)$.

3. $\liminf_{t \to 1} u(t) = \inf_{[0, 1)} u$.

4. $\limsup_{t \to 1} u(t) = \sup_{[0, 1)} u$.

If u is unbounded below, then (1) and (3) are trivial. If u is bounded below, let $\alpha = \inf_{[0, 1)} u$. Now, $u(t) - \alpha \geq 0$ and is not identically zero since u is not constant. By (a) and (b), $u - \alpha = T(u - \alpha) > 0$ on [0, 1), proving (1). Again (3) is now immediate by continuity. If u is unbounded above, then (2) and (4) are trivial. If u is bounded above, the same argument as above applied to $\sup u - u$ shows (2) and (4). What (1) - (4) shows is that for nonconstant C ([0, 1)) fixed points of T either the infimum or supremum may be infinite. They oscillate infinitely many times, thus having unbounded variation on [0, 1), not attaining their supremum or infimum anywhere, but approaching both at 1. ■

One interesting property about the kernel is that the function $\frac{1 + ts}{(1 - ts)^3}$ is an increasing function of t for fixed s. This implies that if $u \geq 0$ then $\int_0^1 \frac{1 + ts}{(1 - ts)^3} u(s)\, ds$ is an increasing function of t, so if u is non-negative then $(Tu)(t) = (1 - t)^2 f(t)$ where f is increasing. In particular, any nonnegative fixed point of T has this property.

Recent research has often been tied to progress in the study of inner divisors on the Bergman space. It is observed that no simple characterisation such as the case known for the Hardy space[2] is possible for the Bergman space. For instance, the union of two zero sequences for the Hardy space is always a zero sequence, but this is not always the case for the Bergman space, as observed by Horowitz[10]. The principal reason for this is that in the Hardy space the intersection of two invariant subspaces is never {0}, unless one of the invariant subspaces is {0} itself. However, the Bergman space does share some features with the Hardy space, for instance, it is shown in [4], [9], a canonical way to factor out zeroes for the Bergman space, using functions analogous to Blaschke products. Associated with the problem of factoring functions in the Bergman space in an effective way these authors defined the concept of inner function suitably modified to this setting.

Definition 9 — A function $G \in L_a^2(D)$ is said to be an inner function (divisor) for $L_a^2(D)$ if $\int_D (|G(z)|^2 - 1)\, g(z)\, dA(z) = 0$ for all $g \in H^\infty$.

This definition of inner function in $L_a^2(D)$ was given by Korenblum and Stessin[11].

Remark 10 : A function $G \in L_a^2(D)$ is called a (norm) expansive multiplier if $\| G \|_{L_a^2} = 1$ and $\| Gg \|_{L_a^2} \geq \| g \|_{L_a^2}$ for all $g \in H^\infty$ If G is an expansive multiplier then G is inner. The converse is also true[4 & 9].

It is shown in [4] and [9] that if $a = (a_j)_{j=1}^{n}$ is a finite set of points in D and $I = \{f \in L_a^2 : f(a_j) = 0, j = 1, ..., n\}$ then there exists an inner function (canonical divisor of a finite zero set) $G \in L_a^2$ such that the following hold: (i) G vanishes at a_j, $j = 1, ..., n$ (ii) $\| G \|_{L^2} = 1$ (iii) $\int_D (|G|^2 - 1)\, u(z)\, dA(z) = 0$, for every function $u \in L_h^1$, the class of harmonic functions in L^1 of the disc (iv) G is equal to a constant plus a linear combination of the Bergman kernel functions $K(z, a_1), \ldots K(z, a_n)$ and some of their derivatives (v) $\int_D |G|^2 |f|^2 dA \geq \int_D |f|^2 dA$ for all $f \in L_a^2$ (vi) $\int_D |G|^2 \psi\, dA \geq \int_D \psi\, dA$ where ψ is an arbitrary bounded positive subharmonic function. Most of the properties also holds when $a = (a_j)_{j=1}^{\infty}$ is an infinite sequence of points in D. Note also that if $G \in L_a^2(D)$ is an inner function (expansive multiplier) then $G \circ \phi_z$ is also an inner function (expansive multiplier) for every $z \in D$ and that an arbitrary invariant subspace of $L_a^2(D)$ contains an expansive multiplier[5].

Given $z \in D$, we define the operator U_z from L_a^2 into L_a^2 as $U_z f(w) = k_z(w) f(\phi_2(w))$. It is easy to show that U_z is a Unitary operator on $L_a^2(D)$ and that $U_z^* = U_z$. We shall now give some sufficient conditions to characterise the set of ϕ in $L^\infty(D)$ for which $\| T_\phi \| = \| \phi \|_\infty$.

Theorem 11 — *Suppose $\phi \in L^\infty(D)$ and that there exists an inner function $G \in L_a^2(D)$ such that* $\left| \int_D \phi(\phi_z(w)) |G(w)| \right|^2 dA(w) | \geq |\phi(z)|$ *for all $z \in D$. Then $\| T_\phi \| = \| \phi \|_\infty$.*

PROOF : The operator T_ϕ is bounded since $\phi \in L^\infty(D)$. Now $\| T_\phi \| = \sup_{f \in L_a^2, \| f \| = 1} |\langle T_\phi f, f \rangle|$

$$\geq \sup_{z \in D, f \in L_a^2, \| f \| = 1} |\langle T_\phi U_z f, U_z f \rangle| \geq \sup_{z \in D} |\langle T_\phi U_z G, U_z G \rangle|$$

$$\sup_{z \in D} |\int_D \phi(*w) |U_z G(w)|^2 dA(w)| = \sup_{z \in D} \left| \int_D \phi(w) \right| G(\phi_z(w))|^2 |k_z(w)|^2 dA(w)|$$

$$= \sup_{z \in D} \left| \int_D \phi(\phi_z(w)) |G(w)|^2 dA(w) \right|$$

$$\geq \left(\sup_{z \in D} |\phi(\phi_z(0))| \right) = \sup_{z \in D} |\phi(z)| = \| \phi \|_\infty.$$

Thus $\| T_\phi \| \geq \| \phi \|_\infty$ and the result follows. ■

Let S be the set of $\phi \in L^\infty(D)$ for which there exist an inner function $G \in L_a^2(D)$ such that $\int_D \phi(\phi_z(w)) \mid G(w) \mid^2 dA(w) \geq \mid \phi(z) \mid$ for all $z \in D$. Let $K = \overline{S}$. If $\phi \in K$ then $\| T_\phi \| = \| \phi \|_\infty$.

We shall now prove again Corollary-6 using the operator U_z and the above technique.

Corollary 12 — Let ϕ be a real valued subharmonic function in $L^\infty(D)$. Then $\| T_\phi \| = \| \phi \|_\infty$.

PROOF : For any $a \in D$, let γ_a be the unique geodesic such that $\gamma_a(0) = 0$, $\gamma_a(1) = a$.

Since D is symmetric, there exists a unique $\phi_a \in$ Aut (D) such that $\phi_a \circ \phi_a(z) = z$ and $\gamma_a\left(\frac{1}{2}\right)$ is an isolated fixed point of $\phi_a \cdot \phi_a$ is the geodesic symmetry at $\gamma_a\left(\frac{1}{2}\right)$. In particular, $\phi_a(0) = a$ and $\phi_a(a) = 0$. If $a = 0$, then we have $\phi_a(z) = -z$ for all z in D. We denote by m_a the geodesic midpoint $\gamma_a\left(\frac{1}{2}\right)$ of 0 and a. In fact if $a \in D$, then $m_a = \frac{1 - \sqrt{1 - \mid a \mid^2}}{\mid a \mid^2} a$. Again note that $k_{m_a}(\phi_a(z))\, k_a(z) = k_{m_a}(z)$ for all a and z in D where k_a is the normalised reproducing kernel for the Hilbert space $L_a^2(D)$ (for proof see [16]).

Now
$$\| T_\phi \| \geq \sup_{a \in D} \mid \langle T_\phi U_a k_{m_a}, U_a k_{m_a} \rangle \mid = \sup_{a \in D} \left| \int_D \phi(w) \mid k_{m_a}(w) \right|^2 dA(w)$$

$$= \sup_{a \in D} \left| \int_D \phi(\phi_{m_a}(w))\, dA(w) \right| \geq \sup_{a \in D} \mid \phi(\phi_{m_a}(0)) \mid = \sup_{a \in D} \mid \phi(m_a) \mid = \| \phi \|_\infty. \quad ■$$

Corollary 13 — Suppose $\phi \in L_h^1(D, dA)$. Then T_ϕ is bounded if and only if ϕ is bounded and in this case $\| T_\phi \| = \| \phi \|_\infty$.

PROOF : Since $\phi \in L_h^1$ therefore, $\tilde{\phi} = \phi$. Now if ϕ is bounded then $\| T_\phi \| \leq \| \phi \|_\infty$. If T_ϕ is bounded then $\mid \phi(z) \mid = \mid \tilde{\phi}(z) \mid = \mid \langle T_\phi k_z, k_z \rangle \mid \leq \| T_\phi \|$ for all $z \in D$. Hence, the result follows. ■

The following Corollary demonstrates how to obtain the inner function G that satisfies the condition of Theorem 11 when ϕ is a subharmonic function of the form $\phi = \mid f \mid^2$ where $f \in H^\infty$. The result also follows directly from Corollary 12.

Corollary 14 — Suppose $\phi \in L^\infty(D)$ and $\phi = \mid f \mid^2$ where $f \in H^\infty$. Then $\| T_\phi \| = \| \phi \|_\infty$.

PROOF : Since $f \in L_a^2$ therefore ϕ is subharmonic. Now observe there exists $g \in L_a^2(D)$ such that $\langle T_\phi g, g \rangle \neq 0$ for otherwise $T_\phi = 0$ and therefore $\phi = 0$ and there is nothing to prove. Let S be the closure of the set of all polynomial multiples of the function $g \in L_a^2(D)$. Note S is an invariant subspace of $L_a^2(D)$ with respect to the Bergman shift operator [5]. Let G be an expansive multiplier in S (cf. [5]).

Now
$$\| T_\phi \| \geq \sup_{z \in D} | \langle \phi U_z G, U_z G \rangle | = \sup_{z \in D} \left| \int_D \phi(\phi_z(w)) | G(w) |^2 dA(w) \right|$$

$$\geq \sup_{z \in D} \int_D | \phi(\phi_z(w)) dA(w) |$$

(because ϕ is harmonic, ϕ_z is conformal for all $z \in D$ and $\int_D |G|^2 \psi dA \geq \int_D \psi dA$ for all bounded subharmonic functions ψ.)

$$\geq \sup_{z \in D} | \phi(\phi_z(0)) | = \sup_{z \in D} | \phi(z) |. \quad \blacksquare$$

If in the above Corollary we replace $f \in H^\infty$ by $f \in L_h^2$ the result again holds.

Let $\Gamma(z, \zeta)$ be the biharmonic Green function of D. That is,

$$\Gamma(z, \zeta) = \frac{1}{16\pi} \left\{ |z - \zeta|^2 \log \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|^2 + (1 - |z|^2)(1 - |\zeta|^2) \right\}.$$

Note $\Gamma(z, \zeta) > 0$ in D.

The following Corollary gives a condition for the equivalence of the operator norm of T_ϕ and the essential norm of the symbol ϕ.

Corollary 15 — If $\phi \in L^\infty$ is such that $\phi \perp h^\infty$ and $\left| \int_D \phi(\zeta) \Delta_\zeta \Gamma(z, \zeta) dA(\zeta) \right| \geq \frac{c}{8} \| \phi \|_\infty$ for some c such that $0 < c \leq 1$ then $\| T_\phi \| \geq c \| \phi \|_\infty$.

PROOF : Let $C_0^\infty(D)$ be the set of infinitely differentiable functions with compact support in D. Since $\phi \perp h^\infty$, there is [5] a sequence of functions $\phi_j \in C_0^\infty(D)$ such that

(a) $\| \Delta \phi_j - \phi \|_1 \to 0$

(b) $\| \phi_j - H \|_1 \to 0$ as $j \to \infty$,

where
$$H(z) = \int_D \phi(\zeta) \Delta_\zeta \Gamma(z, \zeta) dA(\zeta).$$

Now

$$\| T_\phi \| \geq \sup_{f \in L_a^2, \| f \| = 1} \left| \int_D \phi(w) | f(w) |^2 \, dA(w) \right|$$

$$= \sup_{f \in L_a^2, \| f \| = 1} \left| \lim_{j \to \infty} \int_D (\Delta \phi_j)(w) | f(w) |^2 \, dA(w) \right|$$

$$= \sup_{f \in L_a^2, \| f \| = 1} \left| \lim_{j \to \infty} \int_D \phi_j(w) \Delta (| f(w) |^2) \, dA(w) \right|$$

$$\geq \left| \int_D H(w) \Delta (| \sqrt{2}\, w |^2) \, dA(w) \right| = 8 \left| \int_D \int_D \phi(\zeta) \Delta_\zeta \Gamma(w, \zeta) \, dA(\zeta) \, dA(w) \right|$$

$$\geq c \| \phi \|_\infty .$$

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