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WEYL'S THEOREM FOR OPERATORS WITH A GROWTH CONDITION AND DUNFORD'S PROPERTY (C)

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Dedicated to Professor Yong Tae Kim on his 65th birthday

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K. Oberai showed that Weyl's theorem holds for spectral operators of finite type on Banach space. In this paper we generalize this to a class of operators with a growth condition and Dunford's property (*C*).

Key Words : Dunford's property (*C*), a growth condition, Weyl's theorem

INTRODUCTION

Let X be an infinite dimensional complex Banach space and let $B(X)$ be the set of all bounded linear operators acting on a Banach space X. An operator $T \in B(X)$ is called *Fredholm* if the range of T, denoted by $R(T)$, is closed, and the kernel of T, denoted by $N(T)$, and $X/R(T)$ are both finite dimensional. If T is Fredholm, then the *index* of T is defined by

$$\text{ind}\,(T) = \dim N\,(T) - \dim X/R(T)$$

and a Fredholm operator with index zero is called *Weyl* ([8], [9], [10]). If $T \in B(X)$, then we shall denote $\sigma(T)$, $\sigma_p(T)$, $\pi_0(T)$, iso $\sigma(T)$ by the spectrum of T, the set of all eigenvalues of T, the set of all eigenvalues of finite multiplicity of T, and the set of all isolated points of $\sigma(T)$, respectively. We write

$$\pi_{00}(T) = \pi_0(T) \cap iso\ \sigma(T)$$

for the set of all isolated eigenvalues of finite multiplicity of T. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.$$

Following Coburn ([3]) we say that *Weyl's theorem holds for T* if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

Coburn ([3, Theorem (3.1)] showed that Weyl's theorem holds for hyponormal operators and Toeplitz operators. After Coburn's results, many authors found several classes of operators acting on a Hilbert space for which Weyl's theorem holds (see [1], [2], [7], [10], [15] for further references). On the other hand, Gustafson ([7]), Harte and Lee ([10]), and Schechter ([15]) generally considered various conditions that Wely's theorem holds for a Banach space operator. In particular, Oberai ([14]) showed that Weyl's theorem holds for a spectral operator of finite type on a Banach space.

In this paper we extend this to operators which satisfy Dunford's property (C) together with a growth condition G_m. We recall that $T \in B(X)$ satisfies the growth condition G_m provided

$$\sup_{\lambda \notin \sigma(T)} \| (T-\lambda I)^{-1} \| \operatorname{dist}(\lambda, \sigma(T))^m < \infty; \qquad \text{... (1.1)}$$

for example [5] spectral operators of type $m - 1$ satisfy G_m.

We also need some local spectral theory: we say that $T \in B(X)$ has the *single valued extension property* if there is implication, for arbitrary open sets $U \subseteq \mathbb{C}$ and holomorphic functions $f: U \to X$,

$$(T-zI) f(z) = 0 \text{ on } U \Rightarrow f(z) = 0 \text{ on } U. \qquad \text{... (1.2)}$$

If this holds for a neighbourhood Y of $\lambda \in \mathbb{C}$ we say that T has SVEP at λ. When $F \subseteq \mathbb{C}$ is a closed set we write

$$\chi_T(F) = \{y \in X : y = (T-zI) f(z) \text{ for some } f \in \text{Holo } (F, X)\};$$

then Dunford's property (C) is the condition that

$$F = cl\, F \subseteq \mathbb{C} \Rightarrow \chi_T(F) = cl\, \chi_T(F) \subseteq X. \qquad \text{... (1.3)}$$

It is familiar [13] that Dunford's property (C) implies SVEP : (1.3) $\Rightarrow$ (1.2). When $F = \{\lambda\}$ is a singleton we write $\chi_T(F) = \chi_T(\lambda)$, and recall

$$\chi_T(\lambda) = \{x \in X : \| (T-\lambda I)^n x \|^{1/n} \to 0 \text{ as } n \to \infty\} \qquad \text{... (1.4)}$$

is a sort of generalized kernel of $T - \lambda I$; we also write

$$K_T(\lambda) = \{y \in X : \lambda \notin \sigma_T(X)\} \qquad \text{... (1.5)}$$

for what Mbekhta has called the *coeur analytique* of $T - \lambda I$ [16]. This can be characterized (13).

$$K_T(\lambda) = \bigcup \{Y \subseteq X : TY \subseteq Y = TY\} \qquad \text{... (1.6)}$$

as the union of the closed invariant subspaces for T on which the restriction of T is surjective. Our main result is

Theorem 1 — *If $T \in B(X)$ has Dunford's property (C) and also satisfies the growth condition G_m then Weyl's theorem holds for T.*

We actually prove a more general result, involving a pair of operators S and T. We recall that $T \in B(X)$ is said to be a *quasi-affine transform* of $S \in B(Y)$, written $T < S$, if S and T are "intertwined" by an operator $V \in B(X, Y)$ which is both one-one and dense; more generally we shall write $T <^i S$ to mean that there is $V \in B(X, Y)$ for which

$$V \text{ is one-one and } SV = VT.$$

Now what we prove is

Theorem 2 — *If* $T \in B(X)$ *has Dunford's property (C) while* $S \in B(X)$ *has the growth condition* G_m *then there is implication*

$$T <^i S \text{ and } \sigma(S) \subseteq \sigma(T) \Rightarrow \text{Weyl's theorem holds for } T.$$

Taking $S = T$ gives Theorem 1. Since ([5] Theorem XVI.4.4) spectral operators have Dunford's property (C) it follows that spectral operators which satisfy the growth condition G_m satisfy Weyl's theorem, in particular (Oberai [14]) spectral operators of finite type.

2. Proofs

The proof of Theorem 2 divides neatly into two propositions.

Proposition 1 — If $T \in B(X)$ satisfies Dunford's property (C) then there is inclusion

$$\sigma(T) \setminus \omega(T) \subseteq \pi_{00}(T).$$

PROOF : We recall (eg [9] Theorem 9.8.3) that there is always inclusion

$$\sigma(T) \setminus \omega(T) \subseteq \pi_0(T);$$

thus we must show that if T has Dunford's property (C) then

$$\sigma(T) \setminus \omega(T) \subseteq iso\, \sigma(T).$$

If $\lambda \in \sigma(T) \setminus \omega(T)$ then $T - \lambda I$ is Fredholm and hence [13] provided $\chi_T(\lambda)$ is closed we have

$$\chi_T(\lambda) \text{ finite dimensional,} \qquad \text{... (2.1)}$$

and also [13] provided T has SVEP

$$\chi_T(\lambda) \cap K_T(\lambda) = \{0\}; \qquad \text{... (2.2)}$$

thus if T has Dunford's property (C) then both these hold. We claim that also

$$X = \chi_T(\lambda) + K_T(\lambda); \qquad \text{... (2.3)}$$

for (2.1) guarantees that $\chi_T(\lambda)$ is complemented, so that we can write

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} : \begin{pmatrix} Y \\ \chi_T(\lambda) \end{pmatrix} \rightarrow \begin{pmatrix} Y \\ \chi_T(\lambda) \end{pmatrix},$$

and we claim

$$A\ (Y) = Y.$$

Indeed since $\begin{pmatrix} 0 & 0 \\ B & C \end{pmatrix}$ is a finite rank operator T is Weyl on X iff A is Weyl on Y, while (1.5) the inclusion $N(T-\lambda) \subseteq \chi_T(\lambda)$ guarantees that A is one-one. The characterization (1.6) guarantees that $Y \subseteq K_T(\lambda)$, and hence (2.3). By (2.2) and (2.3) $X = \chi_T(\lambda) \oplus K_T(\lambda)$, which now [16] gives Proposition 1. □

Proposition 2 — Suppose $T \in B(X)$ is arbitrary and that $S \in B(X)$ has the growth condition G_m. If S and T satisfy

$$T <^i S \text{ and } \sigma(S) \subseteq \sigma(T) \quad \text{... (2.4)}$$

then there is inclusion

$$\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T).$$

PROOF : We claim first that if (2.4) holds then

$$\text{iso } \sigma(T) \subseteq iso\ \sigma(S). \quad \text{... (2.5)}$$

This rests on an auxiliary result of Fialkow ([6]) : if $P = P^2 \in B(X)$ is a projection whose range is invariant under T and if F is an open-closed subset of the spectrum of the restriction of T to $P(X)$, then ([6] Theorem 2.5)

$$T <^i S \Rightarrow F \cap \sigma(S) \neq \phi.$$

Taking $F = \{\lambda\} \subseteq$ iso $\sigma(T)$ gives (2.5).

We also need another result of Fialkow: if in particular $P_T(F)$ denotes the *Riesz projection* associated with an open-closed subset $F \subseteq \sigma(T)$, and if F is an open-closed subset both of $\sigma(T)$ and of $\sigma(S)$, then ([6] Lemma 2.1)

$$SV = VT \Rightarrow P_S(F)\, V = VP_T(F). \quad \text{... (2.6)}$$

With the help of this we claim that if S and T satisfy (2.4), and if S has the growth condition G_m, then

$$(T - \lambda I)\, P_T(\lambda) = 0. \quad \text{... (2.7)}$$

Indeed if $\lambda \in$ iso $\sigma(S)$ and if S has G_m then the same argument ([10] Theorem 14) as in the case $m = 1$ gives the norm of the restriction of S to the range of $P_S(\lambda)$ equal to zero: now by (2.6)

$$0 = (S - \lambda I)\, P_S(\lambda)\, V = (S - \lambda I)\, VP_T(\lambda) = V(T - \lambda I)\, P_T(\lambda),$$

giving (2.7) since V is one-one. But now the operator $T - \lambda I = 0 \oplus T_1$ is Weyl, being the direct sum of an infinite dimensional zero and an invertible. □

Now, we conclude with some examples of Hilbert space operators :

(a) Wadhwa ([17, Theorem 2.26]) proved that operators with the growth condition G_m whose spectrum lie in a rectifiable Jordan curve satisfies Dunford's property (C). Thus Weyl's theorem holds for operators with the growth condition G_m whose spectrum lie in a rectifiable Jordan curve.

(b) If T is *subdecomposable* (for the definition and details, see [12]), it has Dunford's property (C). Thus Weyl's theorem holds for subdecomposable operators with the growth condition G_m.

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