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OPTIMUM DRAG PROFILE IN AXISYMMETRIC STOKES FLOW

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In this paper, the problem of drag optimization of an axis-symmetric body placed in a uniform stream parallel to the axis of symmetry when the flow is governed by Stokes equations have been tackled. Here we take up a class of bodies to be of given maximum cross-section with fore and aft symmetry about this section. We consider following two cases

case I constant volume constraint and
case II constant meridional cross-section constraint

The possible shape for the stationary value drag has been obtained by making use of method of extremals [Fox (1950), Elsgolc (1962), Sagan (1969)]

Key Words : Optimum Drag; Stokes Flow; Axisymmetric Body

It has been found that the body possesses conical front and rear ends. The value of the drag has also been calculated for the maximum drag body and compared with some known values.

1. INTRODUCTION

Determination of optimum profile in fluid motion is a classical problem. Minimization of wave resistance has its commercial importance in naval architect. The flow then is taken as potential. For viscous drag reduction in rocket, aircraft, ship and automobile engineering, it is the boundary layer theory that matters. But for slow motion such as of acquatic animals, locomtion of microorganisms, settling of tiny particles in chemical engineering, the optimum drag profile is governed by Stokes equations.

Although Stokes equations being linear are simple in appearance, yet the analytical solution of the problem of determining the body profile which extremizes the drag is very complex, and has been defying a complete answer. However, mention may be made of the remarkable paper of Pironneau (1973) wherein he obtained the first order necessary optimality condition for the shape of minimum drag problem. But for determining the profile a very accurate solution of Stokes system is needed which is not easy. On the basis of the above study, a numerical computation of the optimum profile in Stokes flow was made by Bourot (1974). In another interesting paper Taseli *et al.* (1989) employed geometrical shape function to tackle the problem by using simplex method.

The method attempted here is to exploit a simple Stokes drag formula dependent on the geometry of the body and which is derived by us in paper (Datta and Srivastava 1999). The analysis although very simple is subject to the limitations imposed by the applicability of the formula which

can at the most be taken as first step for tackling an intricate problem. It may be mentioned that our class of axial bodies is also subject to the constraint of a prescribed mid-cross section area, in other words the semi transverse axis is taken to have a fixed value b. This implies that corresponding to different constraint volumes V_m, we will have different shapes and hence different semi-axis a_m.

2. CASE I : STATIONARY DRAG PROFILE; FIXED VOLUME CONSTRAINT

In our paper (Datta and Srivastava, 1999), we have advanced the following simple formula for evaluating the drag on an axi-symmetric body placed in axial uniform flow

$$F_x = (1/2)\,(\lambda b^2)/h_m, \quad \lambda = 6\,\pi\,\mu\,U. \qquad \ldots (2.1)$$

Here, the semi-transverse axis (b) is kept fixed and h_m is given by

$$h_m = \frac{3}{4}\int_0^{\pi/2} R \sin^3 \alpha\, d\,\alpha, \qquad \ldots (2.2)$$

and the profile is taken to preserve fore- aft symmetry. The subscript m in h_m and later on a_m is seen to refer to the volume of the body.

It will be found convenient to work in cartesian co-ordinates. Therefore, referring to the figure 1, for the profile geometry, we have

$$y = R \sin \alpha, \tan \alpha = -1/(dy/dx) = -dx/dy = -x'. \qquad \ldots (2.3)$$

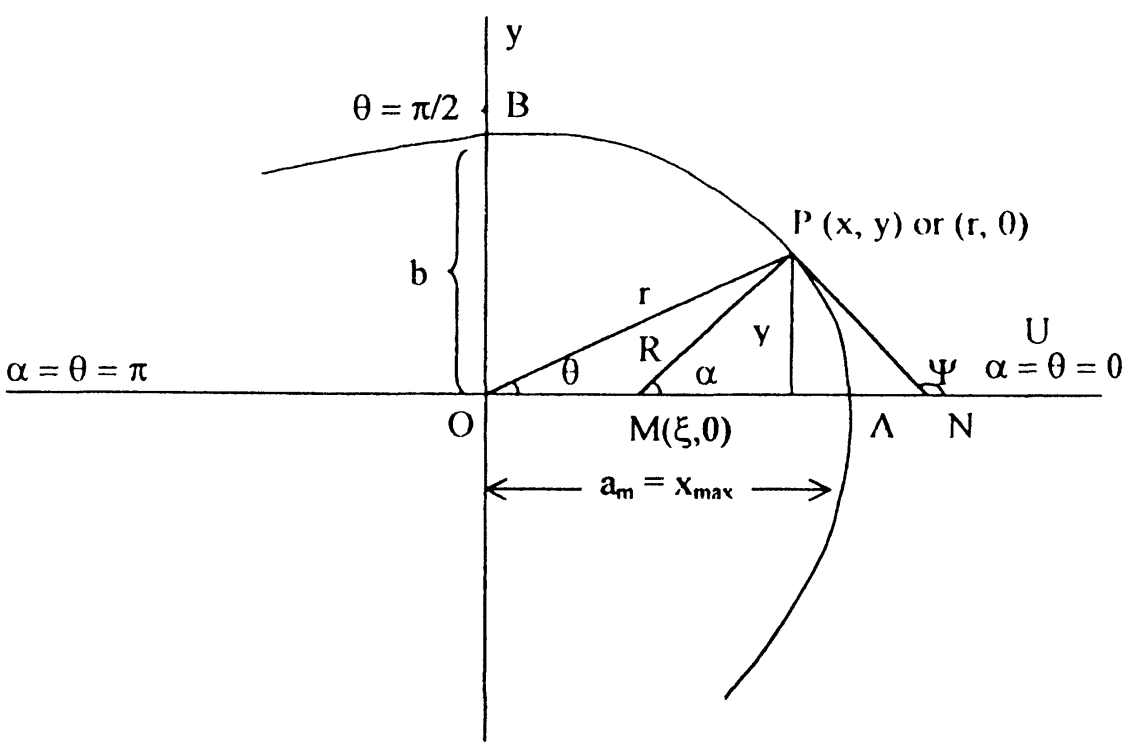

FIG. 1. Geometry of axially symmetric body

Using above transformation, we may express (2.2) as

$$h_m = -\frac{3}{4}\int_0^{a_m} \frac{yy''}{\left\{1+y'^2\right\}^2}\,dx,$$

where 2 a_m represents the axis of the body and dashes represents derivatives with respect to x.

In the sequel, it will be found simpler to work with y as the independent and x as the dependent variable. Thus, h_m assumes the form

$$h_m = -\frac{3}{4}\int_0^{b} \frac{yx'^2 x''}{(1+x'^2)^2}\,dy, \qquad \ldots (2.5)$$

where dashes represents derivatives with respect to y.

Next, the fixed volume V_m of the body is expressible as

$$V_m = 4\pi \int_0^{a_m} y^2\,dx = -4\pi \int_0^{b} y^2 x'\,dy \qquad \ldots (2.6)$$

Now, we have to extremize h_m as given by (2.4) subject to the constant value of V_m as given in (2.6). This may be achieved by making use of Euler-Lagrange equation (Forsyth, 1927).

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \qquad \ldots (2.7)$$

where $$F(x, y, y', y'') = -\frac{3}{4}\left[\frac{yy''}{(1+y'^2)^2} + \frac{m}{4b^2}y^2\right],$$

$m/4b^2$ being Langrange's multiplier. When expressed in terms of y as independent variables we will have corresponding to (2.5) and (2.6)

$$f(y, x, x', x'') = -\frac{3}{4}\left[\frac{yx'^2 x''}{(1+x'^2)^2} - \frac{m}{4b^2}y^2 x'\right], \qquad \ldots (2.9)$$

the Euler-Lagrange equation, now being

$$f_x - \frac{d}{dy}f_{x'} + \frac{d^2}{dy^2}f_{x''} = 0. \qquad \ldots (2.10)$$

Thus, in general, will lead to a fourth order differential equation, there by needing four boundary conditions, two at the end A (y = 0) and two at the end B (y = b). Since, we need to have tangent horizontal at B, the latter two conditions are

$$x = 0,\ x' \to -\infty \text{ (i.e., } dy/dx = 0) \text{ at } y = b, \qquad \ldots (2.11)$$

since x decreases with y, we associate negative sign with infinity.

Now, from (2.9), we have

$$\begin{cases} f_x = \dfrac{\partial f}{\partial x} = 0, \\ f_{x'} = \dfrac{\partial f}{\partial x'} = -\dfrac{3}{4}\left[y \dfrac{d}{dy}\left\{ \dfrac{x'^2}{(1+x'^2)^2} \right\} - \dfrac{m}{4b^2} y^2 \right], \\ f_{x''} = \dfrac{\partial f}{\partial x''} = -\dfrac{3}{4}\left[\dfrac{y x'^2}{(1+x^2)^2} \right] \end{cases}$$

Taking, $f_x = 0$ in eq. (2.10), and integrating, we have

$$\frac{d}{dy} f_{x''} - f_{x'} = \text{constant}$$

Substituting the values of $f_{x'}$ and $f_{x''}$ from (2.12), the above equation immediately provides

$$\frac{x'^2}{(1+x'^2)^2} = \frac{m}{4b^2}(k^2 - y^2)$$

or
$$\frac{x'}{1+x'^2} = -\frac{\sqrt{m}}{2b}\sqrt{k^2 - y^2} \qquad \ldots (2.14)$$

where negative sign is retained since x decreases with y. Here k is constant of the integration. The value $k^2 = b^2$ follows by using the second of boundary condition (2.11), viz., $x' \to -\infty$ at $y = b$.

Here, it may be noted that instead of the expected third order differential equation, eq. (2.14) turns out to be only a first order non-linear differential equation. Thus, it is to be taken as a degenerate case and hence we drop the two boundary conditions at the end A.

Eq. (2.14) may be solved to yield

$$x' = \frac{dx}{dy} = -\frac{1}{\sqrt{m}(b^2 - y^2)}\left[b + \left\{ b^2 - m(b^2 - y^2) \right\}^{1/2} \right]. \qquad \ldots (2.15)$$

Also, the negative root before the curly brackets is dropped because otherwise at $y = b$, we will have the untenable value $x' = 0$. Eq. (2.15) is seen to satisfy the condition $x' \to -\infty$ at $y = b$.

Next, integrating (2.15), and using first of the boundary conditions (2.11), we get

$$x = \frac{b}{\sqrt{m}}\left[\sin^{-1}\sqrt{1 - y^2/b^2} + E\left(\sin^{-1}\sqrt{1 - y^2/b^2} / \sin^{-1}\sqrt{m}\right) \right] 0 \le m \le 1$$

where
$$E(\phi \sin^{-1}\sqrt{m}) = \int_0^{\phi} \frac{(1 - mt^2)^{1/2}}{(1 - t^2)^{1/2}}\, dt \qquad \ldots (2.17)$$

is elliptic function of second kind (Abramowitz, 1964). Also, the maximum axial length $2a_m$ follows from (2.16) when $y = 0$ and is given by

$$a_m = \frac{b}{\sqrt{m}}\left[\frac{\pi}{2} + E(m)\right], \qquad \text{... (2.18)}$$

where $$E(m) = \int_0^1 \sqrt{\frac{1-mt^2}{1-t^2}}\, dt$$

is complete elliptic integral of the second kind (Abramowitz, 1964).

It may be checked that had we started with x as the in dependent variable and used the Euler-Lagrange equation (2.7), we would have obtained the same solution (2.16).

It still remains to deterine the parameter m. This follows by imposing the constant volume constraint. Thus, on substituting the values of x and x' (2.16) and (2.15) respectively, we have

$$V_m = -4\pi \int_0^b y^2 x'\, dy$$

$$= -\frac{4\pi}{\sqrt{m}} \int_0^b y^2 \left[\frac{1}{\sqrt{b^2-y^2}}\left\{b + (b^2 - m(b^2-y^2))^{1/2}\right\}\right] dy,$$

$$= \frac{4\pi b^3}{\sqrt{m}}[J_1 + J_2], \qquad \text{... (2.20)}$$

where, with $y = bt$

$$J_1 = \int_0^1 \frac{t^2\, dt}{\sqrt{1-t^2}} = \pi/4 \qquad \text{... (2.21)}$$

and $$J_2 = \int_0^1 \sqrt{1-t^2}\sqrt{1-mt^2}\, dt$$

Now, we evaluate the integral (2.22) by integrating by parts. We thus get

$$J_2 = \frac{1}{3}E(m) + \frac{1}{3}J_4 \qquad \text{... (2.23)}$$

where $$E(m) = \int_0^1 \sqrt{\frac{1-mt^2}{1-t^2}}\, dt$$

is complete elliptic integral of the second kind and

$$J_4 = \int_0^1 \sqrt{\frac{1-t^2}{1-mt^2}}\, dt = \left(\frac{m-1}{m}\right) K(m) + \frac{1}{m} E(m), \qquad \text{... (2.24)}$$

where
$$K(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \qquad \text{... (2.25)}$$

is the complete elliptic integral of first kind (Abramowitz, 1964).

Hence, eq. (2.23) may be expressed as

$$J_2 = \frac{1}{3}\left[\left(\frac{m+1}{m}\right) E(m) - \left(\frac{1-m}{m}\right) K(m)\right] \qquad \text{... (2.26)}$$

and the volume V_m (2.20) as

$$V_m = \frac{4\pi b^2}{\sqrt{m}}\left[\frac{\pi}{4} + \left(\frac{m+1}{3m}\right) E(m) - \left(\frac{1-m}{3m}\right) K(m)\right]. \qquad \text{... (2.27)}$$

With b taken as constant here, the above result may be expressed in dimensionless form as

$$Q_1(m) = \frac{V_m}{\frac{4}{3}\pi b^3} = \frac{3}{\sqrt{m}}\left[\frac{\pi}{4} + \left(\frac{1+m}{3m}\right) E(m) - \left(\frac{1-m}{3m}\right) K(m)\right]. \qquad \text{... (2.28)}$$

The above equation will provide the value of parameter m for the volume V_m. The variation of $Q_1(m)$ with parameter m is shown in Fig. 2 and Table 1. The maximum drag profiles for various values of m (and hence for various values of the volume V_m) are shown in Fig. 3.

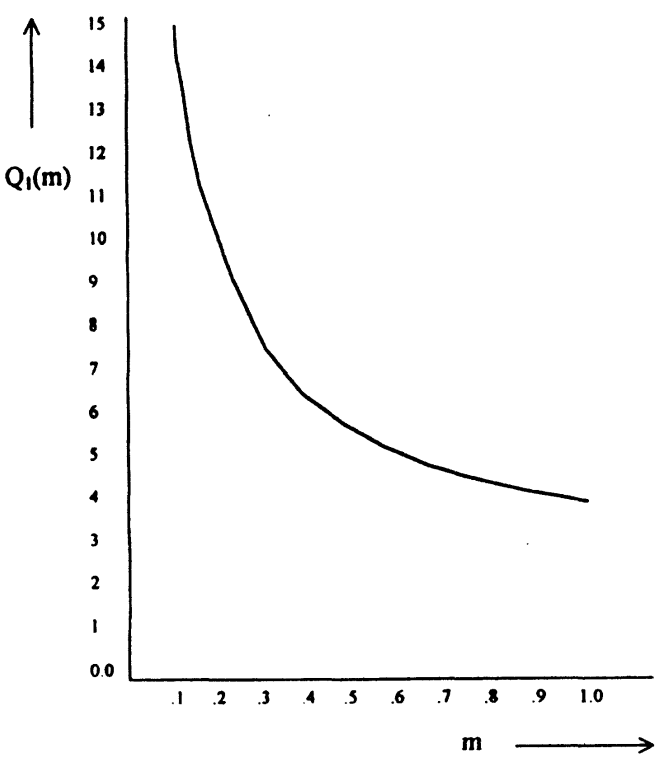

FIG. 2. Variation of $Q_1(m)$ with parameter m.

TABLE 1

m	$Q_1(m) = a/b$	a_m/a	h_m	Maximum drag body		Equivalent volume spheroid	Equivalent volume sphere	2β degrees
				$C_m = F_m/\lambda b$	a_m/b	$C_p = F_p/\lambda b$	$C_s = F_p/\lambda b$	
0.10	14.80	0.66	0.09	5.35	9.80	3.50	2.45	18.44
0.20	10.40	0.65	0.13	3.74	6.80	2.75	2.18	26.58
0.30	8.40	0.65	0.16	3.03	5.50	2.44	2.03	33.22
0.40	7.20	0.65	0.19	2.63	4.70	2.19	1.93	39.24
0.50	6.42	0.64	0.21	2.30	4.10	2.06	1.85	45.00
0.60	5.80	0.63	0.23	2.08	3.70	1.93	1.79	50.78
0.70	5.34	0.61	0.26	1.92	3.30	1.84	1.74	56.80
0.80	4.96	0.61	0.28	1.78	3.07	1.78	1.70	63.44
0.90	4.60	0.61	0.30	1.63	2.82	1.71	1.66	71.58
0.99	4.38	0.60	0.33	1.51	2.59	1.66	1.63	84.26
1.00	4.20	0.60	0.34	1.49	2.57	1.60	1.61	90.00

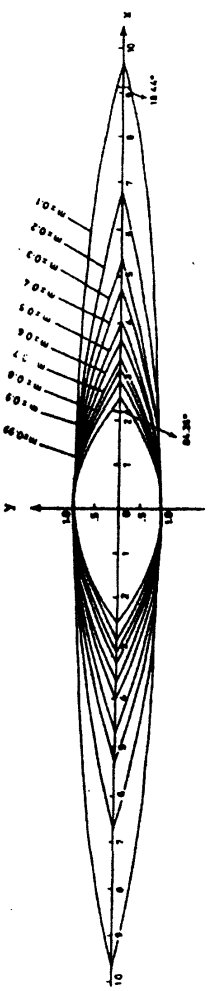

FIG. 3. Maximum drag profile for various values of parameter m (Case I)

3. NECESSARY AND SUFFICIENT CONDITIONS

In the book of Sagan (1969), the necessary and sufficient conditions for minimizing the functional $I\ (y(x))$ have been dependent on the function x, y, y'.

Our functional h_m, to be investigated is seen to depend on a function involving y'' also. Thus, we have as in (2.4)

$$h_m = -\frac{3}{4}\int_0^{a_m} \frac{yy''}{(1+y'^2)}\frac{dx}{2},$$

where a_m is given by (2.18).

We can integrate the above integral by parts to get

$$h_m = \frac{3}{8}\int_0^{a_m}\left[y'\tan^{-1}y' + \frac{y'^2}{1+y'^2}\right]dx$$

where use has been made of the boundary conditions $y' = 0$ at $B\ (0, b)$ and $y = 0$ at $A\ (a_m, 0)$.

Therefore, under constant volume constraint (2.6), we have to study the function

$$F(x, y, y') = \frac{3}{8}\left[y'\tan^{-1}y' + \frac{y'^2}{1+y'^2} - \frac{m}{4b^2}y^2\right], \qquad \ldots (3.1)$$

where we have retained the symbol F, but now it is a function only of x, y, y'. Now, we have

$$\begin{cases} F_y = \frac{3}{8}\left(\frac{-m}{2b^2}y\right) \\ F_{y'} = \frac{3}{8}\left[\tan^{-1}y' + \frac{y'}{(1+y'^2)} + \frac{2y'}{(1+y'^2)^2}\right]. \end{cases}$$

Next, using the values as given in (3.2) in E-L equation and switching over to x as the independent variable, we get

$$\frac{x'^2}{(1+x'^2)} = \frac{m}{4b^2}y^2 = \text{constant}$$

which may be recognized as eq. (2.14). Therefore, as in section 2, we will have the same solution as given by (2.16) and on using the value of x' (2.15), we obtain

$$y' = \frac{dy}{dx} = 1/x' = \frac{\sqrt{m(b^2 - y^2)}}{\left[b + \left\{b^2 - m(b^2 - y^2)\right\}^{1/2}\right]}. \qquad \ldots (3.3)$$

Now, from (3.2), we get

$$F_{y'y'} = \frac{3}{2}\frac{(1-y'^2)}{(1+y'^2)^3}, \qquad \ldots (3.4)$$

and using (3.3), we have on the extremal (2.16)

$$F_{y'y'} = \frac{1}{4b^3}[b^2 - m(b^2-y^2)]^{1/2}[b+(b^2-m(b^2-y^2))^{1/2}]^2. \qquad \ldots (3.5)$$

From (3.3), we find that the maximum value of y'^2 is at $y = 0$.

Next, from (3.4), it follows that

$$Fy'y' > 0$$

when $y'^2 \max < 1.$

Thus, on making use of (3.3) provides us with the condition

$$\frac{m}{(1+\sqrt{1-m})^2} < 1$$

for $F_{y'y'} > 0.$

Since $0 \le m \le 1$, the above inequality holds good except when $m = 1$ and then it is an equality.

Thus, in general, we see that

$$F_{y'y'} > 0. \qquad \ldots (3.7)$$

Taken together with the E-L equation, leads us to conclude that the necessary conditions for h_m to be minimum are being satisfied (Sagan, 1969). Next, we proceed to investigate the sufficiency conditions. For that we have to determine the Jacobi equation and the corresponding Jacobi condition (Sagan, 1969). It will be found convenient to work with y as the independent variable. It may be easily shown that the function $f(y, x, x')$, $x' = dx/dy$, corresponding to $F(x, y, y')$ as given by eq. (3.1) is

$$f(y, x, x') = \frac{3}{8}\left[\frac{\pi}{2} - \frac{x'}{1+x'^2} + \tan^{-1}x' + \frac{m}{2b^2}y^2x'\right]. \qquad \ldots (3.8)$$

The above function may also be obtained from (2.4) by suitable integration by parts. Further it may be checked that this function provides again, through E-L equation, the same differential equation as given by (2.14) leading to the profile given by (2.16).

Now, Jacobi eq. is (1.7)

$$\left[f_{xx} - \frac{d}{dy}(f_{x'x})\right]u - \frac{d}{dy}\left[f_{x'x'}\frac{du}{dy}\right] = 0. \qquad \ldots (3.9)$$

Then, from (3.8), we have

$$\begin{cases} f_{xx} = f_{x'x} = 0 \\ f_{x'x'} = \dfrac{1}{2} \dfrac{\left\{ m(b^2 - y^2) \right\}^{3/2}}{b^3} \left[\dfrac{\sqrt{b^2 - m(b^2 - y^2)}}{b + \sqrt{b^2 - m(b^2 - y^2)}} \right] \end{cases} \quad \ldots (3.10)$$

on the extremal given by (2.16). Thus, Jacobi equation reduces to

$$\frac{d}{dy}\left[f_{x'x'} \frac{du}{dy} \right] = 0, \quad \ldots (3.11)$$

which, using the value of $f_{x'x'}$ as given by (3.10), on integration, yields

$$u(y) = K \int_0^y g(t)\,dt,$$

where $$g(t) = \frac{b^3}{m^{3/2}} \left[1 + \frac{b}{\sqrt{b^2 - m(b^2 - t^2)}} \right] \left[\frac{1}{(b^2 - t^2)^{3/2}} \right].$$

$u(y)$, as given by (3.12), is a solution having a zero at $y = 0$. The integrand $g(t)$ is seen to be a positive function.

Moreover, we find that

$$\frac{d}{dt} g(t) = \frac{t\left[3\left\{ b^2 - m(b^2 - y^2) \right\}^{3/2} + 3b^3 - 4bm(b^2 - y^2) \right]}{m^{3/2} (b^2 - y^2)^{5/2} \left\{ b^2 - m(b^2 - y^2) \right\}^{3/2}}. \quad \ldots (3.13)$$

The condition $$\frac{dg}{dt} \geq 0,$$

is seen to be satisfied when $m \leq 0.75$. Thus, we find that when $m \leq 0.75$m there can not be another zero of $u(y)$ in between $y = 0$ and $y = b$. Hence we conclude that $m \leq 0.75$ is sufficient condition (Sagan, 1969) to ensure that 'h_m' is minimized along the extremal (2.16).

4. Maximum Drag Value and Discussion

The Stokes drag for an axi-symmetric body placed in axial uniform flow is given as (2.1)

$$F_x = \frac{1}{2} \cdot \frac{\lambda b^2}{h_m}.$$

It is clear from the above expression that drag being inversely proportional to the eq. (2.16) provides the profile for maximum drag body.

In order to obtain the value of maximum drag, we now proceed to calculate h_m in terms of parameter m.

Now, using (2.15), the expressions for x'' and $1+x'^2$ may be written as

$$\left\{\begin{array}{l} x''=-\dfrac{b}{\sqrt{m}}\cdot\dfrac{y\left[b+\left\{b^2-m\,(b^2-y^2)\right\}^{1/2}\right]}{(b^2-y^2)^{3/2}\,[b^2-m\,(b^2-y^2)]^{1/2}}, \\ 1+x'^2=\dfrac{2b}{m}\cdot\dfrac{\left[b+\left\{b^2-m\,(b^2-y^2)\right\}^{1/2}\right]}{(b^2-y^2)}. \end{array}\right. \quad \ldots (4.1)$$

Thus, we can express (2.5) as

$$h_m=\frac{3\sqrt{m}}{16\,b}\int_0^b y^2\left[\frac{1}{\sqrt{b^2-y^2}}+\frac{b}{\sqrt{(b^2-y^2)\,(b^2-m\,(b^2-y^2))}}\right]dy. \quad \ldots (4.2)$$

The value of first and second integrals in the above equation are evaluated as

$$\int_0^b \frac{y^2\,dy}{\sqrt{b^2-y^2}}=\frac{b^2\pi}{4},$$

and

$$\int_0^b \frac{y^2\,dy}{\sqrt{(b^2-y^2)\left\{b^2-m\,(b^2-y^2)\right\}}}=\frac{b}{\sqrt{m}}\,[E(m)-(1-m)\,K\,(m)],$$

where $K(m)$ is complete elliptic integral of first kind (Abramowitz, 1964). Hence, we obtain the value of h_m (4.2) on the extremizing curve (2.16)

$$h_m=\frac{3b\sqrt{m}}{16}\left[\frac{\pi}{4}+\frac{1}{m}\left\{E\,(m)-(1-m)\,K(m)\right\}\right]. \quad \ldots (4.3)$$

The numerical values of $Q_1\,(m)$, h_m, F_m (maximum drag), F_p (drag for equivalent volume spheroid), F_s (drag on equivalent volume sphere), a_m (semi-axis for maximum drag profile), a (semi-major axis length for equivalent volume spheroid), 2β (the angle at the front and rear ends of profile) have been given in Table 1 according to the following expressions :

$$Q_1\,(m)=\frac{a}{b}=\frac{3}{\sqrt{m}}\left[\frac{\pi}{4}+\left(\frac{1+m}{3m}\right)E(m)-\left(\frac{1-m}{3m}\right)K\,(m)\right], \quad \ldots (4.4)$$

$$V_m=\frac{4\,\pi b^3}{\sqrt{m}}\left[\frac{\pi}{4}+\left(\frac{1+m}{3m}\right)E\,(m)-\left(\frac{1-m}{3m}\right)K\,(m)\right]$$

$$=\frac{4\,\pi b^3}{3}\,Q_1\,(m),$$

$$h_m=\frac{3b\sqrt{m}}{16}\left[\frac{\pi}{4}+\frac{1}{m}\left\{E(m)-(1-m)\,K\,(m)\right\}\right] \quad (4.3)$$

$$\frac{F_m}{\lambda b} = \frac{8}{3\sqrt{m}} \left[\frac{1}{\frac{\pi}{4} + \frac{1}{m}(E(m) - (1-m)\,K(m))} \right], \qquad \text{... (4.5)}$$

$$\frac{F_p}{\lambda b} = \frac{8}{3} \cdot \frac{a}{b} \cdot \frac{e^3}{\left[-2e + (1+e^2)\,L \right]}, \; L = \log_e \left(\frac{1+e}{1-e} \right), \qquad \text{... (4.6)}$$

where, $a = bQ_1(m),\; e = \sqrt{(1 - 1/Q_1^2)},$

$$\frac{F_s}{\lambda b} = Q_1^{1/3}, \qquad \text{... (4.7)}$$

$$\frac{a_m}{b} = \frac{1}{\sqrt{m}} \left[\frac{\pi}{2} + E(m) \right],$$

$$\beta = \cot^{-1} \left[\frac{1 + \sqrt{1-m}}{\sqrt{m}} \right],$$

where $\lambda = 6\,\pi\,\mu\,U.$

Before taking up the discussion, it will be worth while to observe the following.

Since, b is fixed, it is obvious that we cannot find a sphere of equal volume. The radius of equal volume sphere, $b\,Q_1^{1/3}(m)$, is seen to have minimum value 4.2b (Table 1). The same is true if try to find an equal volume spheroid. For this to be true, we have to satisfy the equality

$$\frac{4}{3}\pi b^2 a_m = V_m = \frac{4}{3}\pi b^3 Q_1(m)$$

where a_m is given by (2.18) and $Q_1(m)$ (2.28).

This leads to the equation

$$(1-m)\,K(m) - E(m) = (\pi/4)\,m.$$

Substituting the values of $K(m)$ and $E(m)$ (Abramowitz, 1964), this reduces to

$$-m \int_0^{\pi/2} \frac{m \sin^2\theta\, d\theta}{(1 - m\sin^2\theta)} = \frac{\pi}{4}\, m,$$

which is not possible except for extreme case of $m = 0$ i.e., infinite volume bodies. Hence no equal volume spheroid exists.

The variation of the non-dimensional volume $Q_1(m) = V_m/(4/3)\,\pi b^3$ is shown graphically in figure 2 and tabulated in Table 1. It is seen that the volume decrease from very large values to the terminating value 4.2 as m increases from value 0 to the value 1. This implies that the volume of the bodies to be considered has at least to be 4.2 times to the volume of the sphere of radius b on the transverse axis.

Further, it is seen that the non-dimensional drag values $F_m/\lambda b, F_p/\lambda b$ and $F_s/\lambda b$ respectively for the maximum drag body, equivalent volume spheroid and equivalent volume sphere all decrease with decreasing volume as it should. It is interesting to observe that while $F_m > F_p$ for $m \leq 0.80$, the inequality is reverse when $m > 0.80$.

The results here may be compared with those given in the paper by Pironneau (1972) where in the profile obtained here is a minimum drag profile. But in this paper, the transverse length at the mid-section is not fixed. The front and rear ends are again conical but the angles now have the fixed value of 120 degrees. In our case, it is seen, from Table 1, that the corresponding angle 2β increase with decreasing volume V_m having the maximum value of 90 degrees. Further, the ratio, while the minimum value of axis ratio a_m/b in our problem is 2.57 (see Table 1 for $m = 1$), the corresponding ratio as calculate from the companion paper of Bourot (1974) is

$$\frac{r(\theta=0)}{r(\theta=\pi/2)} = \frac{2}{2-\frac{2}{\sqrt{3}}+B_1+B_2+B_3+\ldots+B_{10}} = 2.43$$

where $B_1 = -1.847$, $B_2 = 6.337$, $B_3 = -7.212$, $B_4 = -1.964$, $B_5 = 9.365$, $B_6 = 4.137$, $B_7 = -22.31$, $B_8 = 18.32$, $B_9 = -4.502$, $B_{10} = -0.2446$ and never attained by the class of bodies obtained by us.

Thus, our problem is seen to be quite different from that Pironneau and that may be the reason that we are getting a maximum drag profile in place of a minimum one.

5. CASE II STATIONARY DRAG PROFILE; FIXED MERIDIONAL CROSS-SECTION CONSTRAINT

In this section, we repeat the analysis of the secton 2, but now, taking the meridional cross-section to have a fixed value A_m , i.e.

$$A_m = 4\int_0^b y\,x'\,dy, x' = \frac{dx}{dy} \qquad \ldots (5.1)$$

The extremization of h_m, subject to the constant A_m constraint will lead to the consideration of the function

$$f(y, x, x') = \frac{3}{8}\left[\frac{\pi}{2} - \frac{x'}{1+x'^2} + \tan^{-1}x' + \frac{m}{2b}x'y\right], \qquad \ldots (5.2)$$

corresponding to the function (3.8) of the section 2. On differentiating $f(y, x, x')$ partially with respect to x and x' respectively, we get Now, the Euler-Lagrange eq. (2.10) reduces to

$$f_x = 0 \text{ (as } f \text{ is independent of } x) \qquad \ldots (5.3)$$

$$f_{x'} = \frac{\partial f}{\partial x'} = \frac{3}{4}\left[\frac{x'^2}{(1+x'^2)^2} + \frac{my}{4b}\right]. \qquad \ldots (5.4)$$

$$f_{x'} = \text{constant}, \qquad \ldots (5.5)$$

which on using (5.4), provides

$$\frac{x'^2}{(1+x'^2)^2} = \frac{m}{4b}(b-y),$$

where we have made use of the condition that $x' \to -\infty$ at $y = b$. The above equation may be expressed as

$$x'^2 + 2\sqrt{\frac{b}{m(b-y)}}\, x + 1 = 0. \qquad \ldots (5.6)$$

Solving the above quadratic equation, we get

$$\frac{dx}{dy} = x' = -\left[\frac{\sqrt{b} + \sqrt{b - m(b-y)}}{\sqrt{m(b-y)}}\right], \qquad \ldots (5.7)$$

where we have retained the root sign conforming to $x' \to -\infty$ at $y = b$. Integrating (5.7) and using the condition $x = 0$ at $y = b$, we obtain

$$x = 2\sqrt{\frac{b(b-y)}{m}} + \frac{b}{m}\sin^{-1}\sqrt{\frac{m(b-y)}{b}} + \sqrt{(b-y)\left\{\frac{b-m(b-y)}{m}\right\}}. \qquad \ldots (5.8)$$

Again, writing a_m for x at $y = 0$, we get the semi-axis as

$$a_m = x_{\max} = \frac{b}{\sqrt{m}}\left[2 + \sqrt{1-m} + \frac{1}{\sqrt{m}}\sin^{-1}\sqrt{m}\right]. \qquad \ldots (5.9)$$

The value of m is again obtained by the constant area constraint, in this case we set the function $Q_2(m)$, dimensionless constant area as

$$Q_2(m) = \frac{A_m}{\pi b^2}$$

where
$$A_m = 4\int_0^b \frac{y}{\sqrt{m(b-y)}}\left\{\sqrt{b} + \sqrt{b - m(b-y)}\right\} dy$$

$$= \frac{4b^2}{\sqrt{m}}\left[\frac{4}{3} + \frac{1}{\sqrt{m}}\left(1 - \frac{1}{4m}\right)\sin^{-1}\sqrt{m} + \frac{1}{2}\left(1 + \frac{1}{2m}\right)\sqrt{1-m}\right].$$

Thus, we get

$$Q_2(m) = \frac{4}{\pi\sqrt{m}}\left[\frac{4}{3} + \frac{1}{\sqrt{m}}\left(1 - \frac{1}{4m}\right)\sin^{-1}\sqrt{m} + \frac{1}{2}\left(1 + \frac{1}{2m}\right)\sqrt{1-m}\right] \qquad \ldots (5.10)$$

The graph for $Q_2(m)$ against m is depicted in Fig. 4. It is seen that the non-dimensional area $Q_2(m)$ of meridional cross-section decreases with increase in m. Further the extremizing profiles for various values of m are shown in Fig. 5.

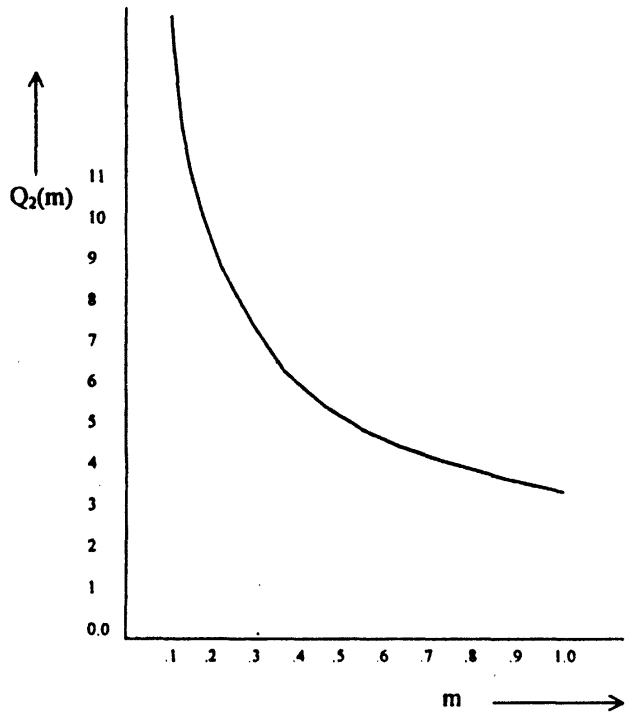

FIG. 4. Variation of $Q_2(m)$ with parameter m

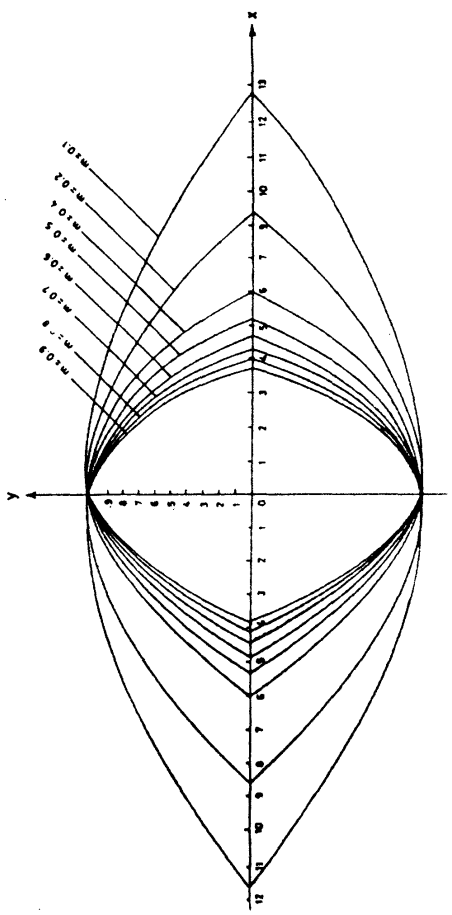

FIG. 5. Maximum drag profile with constant cross-section for various values of parameter m (Case II)

Had we used x as the independent variable, we would have obtained the extremizing function as

$$F(x, y, y') = \frac{3}{8}\left[y' \tan^{-1} y' + \frac{y'^2}{1+y'^2} - \frac{m}{4b} y \right], \qquad \text{... (5.11)}$$

instead of (5.2), where we now have from (5.7)

$$y' = \frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{\sqrt{m(b-y)}}{\sqrt{b}\sqrt{b-m(b-y)}}. \qquad \text{... (5.12)}$$

From (5.11), we get on differentiation again

$$F_{y'y'} = \frac{3}{2}\frac{(1-y'^2)}{1+y'^2)^3}, \qquad \text{... (5.13)}$$

which is same as (3.4) for the constant volume case I. Further, using the value of y' given by (5.12), we have on the extremal

$$F_{y'y'} = \frac{3}{2} \cdot \frac{1}{4b^{3/2}} \cdot \frac{\sqrt{b-m(b-y)}}{[\sqrt{b}+\sqrt{b-m(b-y)}]^{-2}} \qquad \text{... (5.14)}$$

The maximum value of $y_{\max}'^2$ is seen to occure at $y = 0$. Again it is seen from (5.12) that $F y' y' > 0$ when $y_{\max}'^2 < 1$.

Thus, on making use of (5.12), we again get the same condition as (3.6), viz;

$$\frac{m}{(1+\sqrt{1-m})^2} < 1 \text{ for } F_{y'y'} > 0.$$

This implies that the necessary conditions for h_m to be minimum are being satisfied (Sagan, 1969).

Next, we investigate for sufficiency. From (4.2), we obtain

$$\begin{cases} f_{xx} = f_{x'x} = 0 \\ f_{x'x'} = -\frac{3}{2}\frac{x'(x'^2-1)}{(x'^2+1)^3} \\ = \frac{3}{8}\left(\frac{m}{b}\right)^{3/2} \frac{(b-y)^{3/2}\sqrt{b-m(b-y)}}{[\sqrt{b}+\sqrt{b-m(b-y)}]}. \end{cases} \qquad \text{... (5.15)}$$

Now, the Jacobi equation [cf. eqns. (3.9) and (3.11)] will yield

$$u(y) = k \int_0^y g(t)\, dt \qquad \text{... (5.16)}$$

where $$g(t) = \frac{8}{3}\left(\frac{b}{m}\right)^{3/2} \frac{[\sqrt{b} + \sqrt{b - m(b-t)}]}{(b-t)^{3/2}\sqrt{b - m(b-t)}}.$$

$u(y)$, as given by (5.16), is a solution having a zero at $y = 0$, the integrand $g(t)$ is again seen to be a positive function. Therefore $u(y)$ will not have another zero between 0 and b and Jacobi condition [Sagan, 1969] is satisfied.

TABLE 2

m	$Q_2(m) = A_m/\pi b^2$	a_m/a	Maximum drag body		Equivalent volume spheroid	Equivalent volume sphere	2β degrees
			$C_m = F_m/\lambda b = b/2h_m$	a_m/b	$C_p = F_p/\lambda b$	$C_s = F_s/\lambda b = Q_2$	
0.10	10.70	1.17	8.36	12.53	2.76	3.27	18.44
0.20	7.51	1.16	5.90	8.78	2.23	2.74	26.58
0.30	6.09	1.05	4.81	6.40	1.99	2.46	33.22
0.40	5.25	1.16	4.16	6.09	1.83	2.29	39.24
0.50	4.67	1.15	3.70	5.39	1.72	2.16	45.00
0.60	4.23	1.14	3.35	4.85	1.64	2.05	50.78
0.70	3.90	1.14	3.09	4.46	1.57	1.90	56.80
0.80	3.63	1.13	2.87	4.11	1.53	1.70	63.44
0.90	3.37	1.13	2.75	3.82	1.47	1.83	71.58
1.00	3.19	0.85	2.50	3.57	1.44	1.78	90.00

Next, we can obtain the value of x'' from (5.7) as

$$x'' = -\frac{\sqrt{m}}{2(b-y)^{3/2}\sqrt{b - m(b-y)}}. \qquad \ldots (5.18)$$

Substituting the value of x' from (5.7) and of x'' from (5.18) in (2.5), we get

$$h_m = \frac{3}{32} b^2 \sqrt{m}\left[\frac{1}{2m}\left\{\sqrt{1-m} - \left(\frac{1-2m}{\sqrt{m}}\right)\sin^{-1}\sqrt{m}\right\} + \frac{4}{3}\right]. \qquad \ldots (5.19)$$

The values of various associated quantities of interest are shown in table 2. In this case, unlike that for the case of constant volume, we see that $F_m > F_p$ for all values of $0 < m \leq 1$. The shapes of the maximum drag profile for constant cross-section for various values of the parameter m are depicted in figure 5.

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