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ERROR BOUNDS FOR APPROXIMATIONS OF THE DEFINITE INTEGRALS

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Fundamental numerical methods to calculate definite integrals $\int_a^b f(x)\,dx$ include Trapezoidal, Midpoint and Simpson's rule. When calculating definite integrals with these methods, however, people usually apply error approximation rules with strict and specified requirements in numerical analysis. This paper studies integrands which satisfy (1) being differentiable on (a, b) or (2) convex functions with the existence $f'_+(a)$ and $f'_-(b)$, in order to obtain the approximation rule for error bounds.

Key Words: Trapezoidal Rule; Mid-point Rule; Simpson's Rule

INTRODUCTION

In order to evaluate $\int_a^b f(x)\,dx$, using the Fundamental Theorem of Calculus, we need to know the antiderivative of f. Occasionally, however, it is difficult, or even impossible, to find an antiderivative. In virtue of this, we usually evaluate approximate value of the definite integrals instead. The most familiar approximations of the definite integrals are

(i) the Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]\Delta x$$

where $\qquad \Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x;\ i = 1, 2, \ldots, n,$

(ii) the Midpoint Rule

$$\int_a^b f(x)\,dx \approx \left[f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+\ldots+f(\bar{x}_{n-1})+f(\bar{x}_n)\right]\Delta x$$

where $\Delta x=\dfrac{b-a}{n}$ and $\bar{x}_i=a+\left(i-\dfrac{1}{2}\right)\Delta x;\ i=1,2,\ldots,n,$

and (iii) the Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+$$

$$\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]\,\Delta x$$

where n is even, $\Delta x=\dfrac{b-a}{n}$ and $x_i=a+i\,\Delta x;\ i=1,2,\ldots,n.$

The error bounds of Trapezoidal rule, Midpoint rule and Simpson's rule are given as follow :

Let $|f''(x)|\le K$ and $|f^{(4)}(x)|\le M$ for $a\le x\le b$. If E_T, E_M and E_S are the errors in the Trapezoidal, Midpoint and Simpson's Rules, respectively, then

$$|E_T|\le\frac{K(b-a)^3}{12n^2}, \qquad \ldots (1)$$

$$|E_M|\le\frac{K(b-a)^3}{24n^2} \qquad \ldots (2)$$

and

$$|E_S|\le\frac{M(b-a)^5}{180\,n^4}. \qquad \ldots (3)$$

However in many practical situations, the integrands may not be continuously fourth differentiable, among them some are even not continuously twice differentiable so that (1), (2) and (3) cannot be used. One simple example to demonstrate this is $f(x) = (x^3-c)^p$ with $c\neq 0, a<c<b$ and $1 < p < 2$. In this case the antiderivative is difficult to find $\Delta f''(x)$ and $f^{(4)}(x)$ are not bounded in (a, b) but $f'(x)$ is continuous on $[a, b]$. So that (1), (2) and (3) can not be applied to this example.

The aim of this paper is (i) to show that the errors in the Midpoint rule and Simpson's rule satisfy first-order estimate and (ii) to determine the error bounds in the Trapezoidal, Midpoint and simpson's Rules for convex functions.

NOTATIONS AND ERROR BOUNDS OF THE FIRST ORDER ESTIMATE

Let $[a, b]$ be a closed interval in $\boldsymbol{R}$ and let f be a real valued function on $[a, b]$ and let P_n be the regular partition of $[a, b]$ into n subintervals of equal length. Set $\Delta x=\dfrac{b-a}{n}$ and define

$$M_n[a,b] = \sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\Delta x;\ n = 1, 2, 3, \ldots, \qquad \ldots (4)$$

$$T_n[a,b] = \frac{1}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(a + k\Delta x) + f(b)\right]\Delta x;\ n = 1, 2, 3, \ldots, \qquad \ldots (5)$$

and

$$S_n[a,b] = \frac{1}{6}\left[f(a) + 4\sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right) + 2\sum_{k=1}^{n-1} f(a + k\Delta x) + f(b)\right]\Delta x;$$

$n = 1, 2, 3, \ldots$

We note that $T_n[a,b], M_n[a,b]$ and $S_n[a,b]$ are approximations of Trapezoidal rule, Midpoint rule and Simpson's rule, respectively. We also note that $T_n[a,b]$ and $S_n[a,b]$ can be written as follows :

$$T_n[a,b] = \frac{1}{2}\sum_{k=1}^{n} [f(a + (k-1)\Delta x) + f(a + k\Delta x)]\Delta x, \qquad \ldots (6)$$

$$S_n[a,b] = \frac{1}{6}\sum_{k=1}^{n}\left[f(a + (k-1)\Delta x) + 4f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right) + f(a + k\Delta x)\right]\Delta x$$

$$= \frac{1}{3}(T_n[a,b] + 2M_n[a,b]). \qquad \ldots (7)$$

In 1938 Ostrowski[7] proved that if $|f'(x)| \le M;\ a \le x < b$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M, \qquad \ldots (8)$$

and at the same year Iyengar[4] proved that, if $|f'(x)| \le M;\ a \le x < b$, then

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}(f(a) + f(b))\right| \le \frac{M(b-a)^2}{4} - \frac{1}{4M}(f(b) - f(a))^2$$

$$\le \frac{M(b-a)^2}{4}. \qquad .. (9)$$

Using the inequalities (8) and (9) together with

$$\left| \int_a^b f(x\, dx - M_n[a, b] \right|$$

$$\leq \sum_{k=1}^{n} \left| \int_{a+(k-1)\Delta x}^{a+k\Delta x} f(x)\, dx - f\left(a + \left(k - \frac{1}{2} \right) \Delta x \right) \Delta x \right| \qquad \ldots (10)$$

and

$$\left| \int_0^b f(x)\, dx - T_n[a, b] \right|$$

$$\leq \sum_{k=1}^{n} \left| \int_{a+(k-1)\Delta x}^{a+k\Delta x} f(x)\, dx - \frac{\Delta x}{2} (f(a + (k-1)\Delta x) + f(a + k\Delta x)) \right|. \qquad \ldots (11)$$

S. S. Dragomir and S. Wang have proved the following two theorems.

Theorem 2.1 ([3]) — *Let* $f: [a, b] \to \boldsymbol{R}$ *be a differentiable function on* (a, b). *If* $\| f' \|_\infty = \sup_{x \in (a, b)} | f'(x) |$, *then we have*

$$\left| \int_a^b f(x)\, dx - M_n[a, b] \right| \leq \frac{(b-a)^2}{4n} \| f' \|_\infty \qquad \ldots (12)$$

Theorem 2.2 ([2]) — *Let* $f: [a, b] \to \boldsymbol{R}$ *be a differentiable function on* (a, b). *If* $\| f' \|_\infty = \sup_{x \in (a, b)} | f'(x) |$, *then we have*

$$\left| \int_a^b f(x)\, dx - T_n[a, b] \right| \leq \frac{\| f' \|_\infty^2 (b-a)^2 - (f(b) - f(a))^2}{4n \| f' \|_\infty}$$

$$\leq \frac{(b-a)^2}{4n} \| f' \|_\infty. \qquad \ldots (13)$$

For the approximation of Simpson's rule, we have

Theorem 2.3 — *Let* $f: [a, b] \to \boldsymbol{R}$ *be a differentiable function on* (a, b). *If* $\| f' \|_\infty = \sup_{x \in (a, b)} | f'(x) |$, *then we have*

$$\left| \int_a^b f(x)\, dx - S_n[a, b] \right| \leq \frac{(b-a)^2}{4n} \| f' \|_\infty - \frac{1}{12\, n \| f' \|_\infty} (f(b) - f(a))^2$$

$$\leq \frac{(b-a)^2}{4n} \|f'\|_\infty. \qquad \ldots (14)$$

PROOF : It follows from (7), (12) and (13) that

$$\left| \int_a^b f(x)\,dx - S_n[a,b] \right| = \left| \int_a^b f(x)\,dx - \frac{1}{3}(T_n[a,b] + 2M_n[a,b]) \right|$$

$$\leq \frac{1}{3}\left| \int_a^b f(x)\,dx - T_n[a,b] \right| + \frac{2}{3}\left| \int_a^b f(x)\,dx - M_n[a,b] \right|$$

$$\leq \frac{(b-a)^2}{4n} \|f'\|_\infty - \frac{1}{12n\|f'\|_\infty}(f(b)-f(a))^2$$

$$\leq \frac{(b-a)^2}{4n} \|f'\|_\infty.$$

Remark : The estimations in (12), (13) and (14) are of first order accuracy, they depend only on $\|f'\|_\infty$.

THE ERROR ESTIMATION ON CONVEX FUNCTIONS

A real-valued function f defined on an interval $I \subseteq \boldsymbol{R}$ is called convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \qquad \ldots (15)$$

is valid for all $x, y \in I$ and for all real numbers $\lambda \in [0, 1]$. If $f : [a, b] \to \boldsymbol{R}$ is convex, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{1}{2}(f(a)+f(b)). \qquad \ldots (16)$$

The inequality (16) is known as Hermite-Hadamard inequality (see [6]).

In order to estimate the error of convex function, we need the following two lemmas :

Lemma 3.1 — Let $f : [a, b] \to \boldsymbol{R}$ be a convex function and let n be a positive integer. Then we have

(a) $M_n[a,b] \leq \int_a^b f(x)\,dx \leq T_n[a,b]$

(b) $M_n[a,b] \leq M_{2n}[a,b]$

(c) $T_{2n}[a,b] \leq T_n[a,b]$

(d) $T_{2n}[a,b] = \frac{1}{2}(T_n[a,b] + M_n[a,b])$

PROOF : (a), (b) and (c) have proved by S. S. Dragomir (see [1]).

(d) Since

$$M_n[a,b] = \sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\Delta x = \sum_{k=1}^{n} f\left(a + (2k-1)\frac{\Delta x}{2}\right)\Delta x, \quad \ldots (17)$$

it follows from (5) and (17) that

$$\frac{1}{2}\left\{T_n[a,b] + M_n[a,b]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(a + k\,\Delta x) + f(b)\right]\Delta x + \sum_{k=1}^{n} f\left(a + (2k-1)\frac{\Delta x}{2}\right)\Delta x\right\}$$

$$= \frac{1}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f\left(a + 2k\left(\frac{\Delta x}{2}\right)\right) + 2\sum_{k=1}^{n} f\left(a + (2k-1)\frac{\Delta x}{2}\right) + f(b)\right]\left(\frac{\Delta x}{2}\right)$$

$$= \frac{1}{2}\left[f(a) + 2\sum_{k=1}^{2n-1} f\left(a + k\frac{\Delta x}{2}\right) + f(b)\right]\frac{\Delta x}{2}$$

$$= T_{2n}[a,b].$$

This completes the proof.

Let

$$f'_+(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \text{ and } f'_-(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c},$$

Then we have

Lemma 3.2 — Let $f: [a,b] \to \boldsymbol{R}$ be a convex function such that $f'_+(a), f'_-(b)$ exist. Then

$$f'_+(a) \le \frac{f(x) - f(a)}{x - a} \le f'_- x \le f'_+(x) \le \frac{f(b) - f(x)}{b - x} \le f'_-(b), \ \forall\, x \in (a,b).$$

PROOF : It is well-known ([5], p17) that, if $f: [a,b] \to \boldsymbol{R}$ is convex, then $f'_-(x)$ and $f'_+(x)$ exist, and $f'_-(x) \le f'_+(x)$, $\forall\, x \in (a,b)$. Now if $y \in (x,b)$, then there exists $\alpha \in (0,1)$ such that $y = \alpha x + (1-\alpha)\,b$. Since f is convex, it follows that

$$f(y) = f(\alpha x + (1-\alpha) b) \le \alpha f(x) + (1-\alpha) f(b). \quad \dots (18)$$

this implies $f(y) - f(b) \le \alpha (f(x) - f(b))$, so that

$$\frac{f(b)-f(x)}{b-x} \le \frac{f(b)-f(y)}{\alpha(b-x)} = \frac{f(b)-f(y)}{b-y}, \; \forall \; y \in (x, b).$$

Hence $$\frac{f(b)-f(x)}{b-x} \le \lim_{y \to b^-} \frac{f(b)-f(y)}{b-y} = f'_-(b).$$

From (18) we also have

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(x)}{b-x}, \; \forall \; y \in (x, b).$$

This implies

$$f'_+(x) = \lim_{y \to x^+} \frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(x)}{b-x}.$$

Similarly let $y = \alpha a + (1-\alpha) x$, $\alpha \in (0, 1)$, we have

$$f'_+(a) \le \frac{f(x)-f(a)}{x-a} = f'_-(x).$$

This completes the proof.

Now, we are ready to prove the following theorems :

Theorem 3.3 — *Let* $f: [a, b] \to \mathbf{R}$ *be a convex function and assume that both* $m = f'_+(a)$ *and* $M = f'_-(b)$ *exist. Then*

$$0 \le T_n[a, b] - M_n[a, b] \le \frac{(b-a)^2}{4n}(M-m).$$

PROOF : It follows from Lemma 3.1 (a) and the identities (4) and (6), that

$$0 \le T_n[a, b] - M_n[a, b]$$

$$= \sum_{k=1}^{n} \left\{ \frac{1}{2} [f(a+(k-1)\Delta x) + f(a+k\Delta x)] - f\left(a + \left(k - \frac{1}{2}\right)\Delta x\right)\right\} \Delta x$$

$$= \frac{(\Delta x)^2}{4} \sum_{k=1}^{n} \frac{f(a+k\Delta x) - f\left(a + \left(k-\frac{1}{2}\right)\Delta x\right)}{\frac{1}{2}\Delta x}$$

$$- \frac{(\Delta x)^2}{4} \sum_{k=1}^{n} \frac{\left(a+k-\frac{1}{2}\right)\Delta x) - f(a+k-1)\,\Delta x)}{\frac{1}{2}\Delta x}$$

Using Lemma 3.2, we have

$$\frac{f(a+k\,\Delta x) - f\left(a + \left(k-\frac{1}{2}\right)\Delta x\right)}{\frac{1}{2}\Delta x} \le f_-'(a+k\,\Delta x) \le f_-'(b) = M,$$

and

$$\frac{f\left(a + \left(k-\frac{1}{2}\Delta x\right)\right) - f(a+(k-1)\,\Delta x)}{\frac{1}{2}\Delta x} \ge f_+'(a+(k-1)\,\Delta x) \ge f_+'(a) = m,$$

for $k = 1, 2, ..., n$.

Hence $$T_n[a,b] - M_n[a,b] \le \frac{1}{4}(\Delta x)^2 (n(M-m)) = \frac{(b-a)^2}{4n}(M-m).$$

This completes the proof.

Theorem 3.4 — *Let* $f: [a, b] \to \boldsymbol{R}$ *be a convex function and assume that both* $m = f_+'(a)$ *and* $M = f_-'(b)$ *exit. Then we have*

(a) $$0 \le T_n[a,b] - \int_a^b f(x)\,dx \le \frac{(b-a)^2}{4n}(M-m).$$

(b) $$0 \le \int_a^b f(x)\,dx - M_n[a,b] \le \frac{(b-a)^2}{8n}(M-m).$$

(c) $$\left| \int_a^b f(x)dx - S_n[a,b] \right| \le \frac{(b-a)^2}{12n}(M-m).$$

PROOF : (a) By Lemma 3.1 (a) and Theorem 3.3, we have

$$0 \le T_n[a,b] - \int_a^b f(x)\,dx \le T_n[a,b] - M_n[a,b] = \frac{(b-a)^2}{4n}(M-m).$$

(b) By Lemma 3.1 (a) and (c), we have

$$M_n[a,b] \le \int_a^b f(x)\,dx \le T_{2n}[a,b] \le T_n[a,b].$$

It follows from Lemma 3.1 (a), (b) and Theorem 3.3 that

$$0 \le \int_a^b f(x)\,dx - M_n[a,b] \le T_{2n}[a,b] - M_n[a,b]$$

$$= \frac{1}{2}(T_n[a,b] - M_n[a,b])$$

$$\le \frac{(b-a)^2}{8n}(M-m).$$

(c) First, if $\int_a^b f(x)dx \le S_n[a,b]$, then it follows from Lemma 3.1(a), inequality (7) and Theorem 3.3 that

$$0 \le S_n[a,b] - \int_a^b f(x)\,dx \le S_n[a,b] - M_n[a,b]$$

$$= \frac{1}{3}(T_n[a,b] - M_n[a,b])$$

$$= \frac{(b-a)^2}{12n}(M-m).$$

Next, if $S_n[a,b] \le \int_a^b f(x)\,dx$, then it follows from Lemma 3.1 (a), (d), inequality (7) and Theorem 3.3 that

$$0 \le \int_a^b f(x)\,dx - S_n[a,b] \le T_{2n}[a,b] - S_n[a,b]$$

$$= \frac{1}{6}(T_n[a,b] - M_n[a,b])$$

$$= \frac{(b-a)^2}{24n}(M-m).$$

Thus

$$\left| \int_a^b f(x)\,dx - S_n[a,b] \right| \leq \frac{(b-a)^2}{12n}(M-m).$$

Remark : Although our estimations in Theorem 3.4 require that f is a convex function, acutally their accuracies depend only on the difference between $f_-^+(b)$ and $f'_+(a)$. In general there are many convex functions that are not differentiable, so that (1), (2), (3) (12), (13) and (14) can not be used. Thus our results in Theorem 3.4 provide some estimations for convex functions.

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