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FUZZY REAL LINE STRUCTURE AND METRIC SPACE

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The paper is addressed to the introduction of a new notion of fuzzy real line via 'Extension principle' and a cannonical fuzzy metric which is developed using the level representation of fuzzy sets.

Key Words : Fuzzy Number; α-Level Sets; Regular Fuzzy Number; Fuzzy Real Line; Extension Principle and Cannonical Fuzzy Metric

INTRODUCTION

Fuzzy sets are assumed to have its theoretic behaviour almost identical to that of the ordinary sets. Eventually fuzzy set theory is merged into few different directions all based on some deeper examples or constructions.

The paper contains the description of a new notion of fuzzy real line denoted R_L, where L is a complete chain. The fuzzy real line R_L consists of L-fuzzy numbers which fulfil regularity condition. It turns out that R_L has some structure well known for fuzzy numbers, an addition, a multiplication and an ordering that are specially apparent in algebra of fuzzy points. Although the representation of fuzzy points is a major obstacle, the choice of an operator via the 'Extension Principle' is well suited in our present construction.

The paper therefore, addressed to the introduction of a new notion of fuzzy real line and a cannonical fuzzy metric which is defined using the level representation of fuzzy sets.

THE FUZZY REAL LINE

An L-fuzzy number or simply a fuzzy number is a fuzzy set on the *real axis R* i.e., a mapping $\lambda : R \to [0, 1] = L$, associating with each real number t, its grade of membership $\lambda(t)$.

A fuzzy number λ is called convex if

$$\lambda(t) \geq \min(\lambda(s), \lambda(r)), s \leq t \leq r$$

Each set $[\lambda]_\alpha = \{t \mid \lambda(t) \geq \alpha\}, 0 < \alpha \leq 1$ is called α-level set of λ and λ is convex if and only if its α-level set $[\lambda]_\alpha$ is a convex set in IRI (Zadeh[2]).

If there exists a $t_0 \in R$ such that $\lambda(t_0) = 1$, then λ is called normal.

A fuzzy number λ will be called upper semi continuous provided for all $t \in R$ and $\alpha \in L$ with $\lambda(t) < \alpha$, there is a $\delta > 0$ such that $|s - t| \leq \delta \Rightarrow \lambda(s) < \alpha$.

Further λ will be called regular if λ is convex, normal as well as upper semi continuous and each level of λ is bounded.

The level sets of a regular fuzzy number λ are closed finite intervals.

$$\lambda\alpha = [\lambda\alpha_1, \lambda\alpha_2], \text{ where } \lambda\alpha_1 = -\infty \text{ and } \lambda\alpha_2 = \infty \text{ are admissible values.}$$

Let R_L denote the set of all regular L-fuzzy numbers. Then R_L will be called the L-fuzzy real line.

Since each $\lambda \in R$ can be considered as a fuzzy number $\bar{\lambda}$ defined by

$$\bar{\lambda}(t) = 1 \ (t = \lambda) \ = \ 0 \ (t \neq \lambda),$$

the real numbers can be embedded in R_L. These fuzzy numbers are called crisp.

A fuzzy number λ is called non negative if $\lambda(t) = 0$ for all $t < 0$.

The equality of fuzzy numbers λ and μ is defined by

$$\lambda = \mu \text{ if and only if } \lambda(t) = \mu(t) \text{ for all } t \in R.$$

The arithmetic operations $+$, $-$, $\cdot$ and $/$ on $R_L \times R_L$ are defined (Mizumoto and Tanaka [4]) by

$$(\lambda + \mu)(t) = \bigvee_{\Sigma \in P} [\min(\lambda/s), \mu(t - s))], t \in R$$

$$(\lambda - \mu)(t) = \bigvee_{\Sigma \in P} [\min(\lambda(s), \mu(s - t))], t \in R$$

$$(\lambda\mu)(t) = \bigvee_{\substack{\Sigma \in P \\ \Sigma \neq P \neq 0}} [\min(\lambda(s), \mu(t/s))], t \in R$$

$$(\lambda/\mu)(t) = \bigvee_{\Sigma \in P} [\min(\lambda(ts), \mu(s))], t \in R$$

These definitions are special cases of extension principle of Zadeh (Zadeh [3]). The additive and multiplicative identities in R_L and $\bar{0}$ and $\bar{1}$, where

$$\tilde{0}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\tilde{1}(t) = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{if } t \neq 1 \end{cases}$$

Let $-\mu$ be defined as $\tilde{0} - \mu$

It follows by above definition that

$$(-\mu)(t) = \mu(-t) \text{ for all } t \in R \text{ and } \lambda - \mu = \lambda + (-\mu),\ -(\lambda + \mu) = (-\lambda) + (-\mu).$$

The absolute value $|\lambda|$ of $\lambda \in R_L$ is defined by

$$|\lambda|(t) = \begin{cases} \max(\lambda(t)\, \lambda(-t)), & t \geq 0 \\ = 0 \text{ if } t < 0 \end{cases}$$

The equations $a + \lambda = \tilde{0}$ and $a \cdot \lambda = \tilde{1}$ have unique solutions if and only if a is crips (Kaleva and Seikkala[8])

The operations + and $\cdot$ are associative and commutative with the identities $\tilde{0}$ and $\tilde{1}$ respectively. The level sets of sum, difference and product fulfil the following conditions

$$[\lambda + \mu]_\alpha = [\lambda]_\alpha + [\mu]_\alpha$$

$$[\lambda - \mu]_\alpha = [\lambda]_\alpha - [\mu]_\alpha$$

In general, it is not obvious that for $f: X \times X \to Z$ and λ, μ fuzzy subsets of X and Y respectively, we would always have

$f[(\lambda, \mu)]_\alpha = f([\lambda]_\alpha, [\mu]_\alpha)$ for all $\alpha \in L$. However it may be shown that if L = [0, 1] and $f: R \times R \to R$ be continuous, then $f[(\lambda, \mu)]_\alpha = f([\lambda]_\alpha, [\mu]_\alpha)$ for all $\alpha \in L$ and all upper semi continuus L-fuzzy sets λ and μ on R.

We define a partial ordering '$\leq$' on R_L by $\lambda \leq \mu$ iff $\lambda\, \alpha_i \leq \mu\, \alpha_i$ for all $\alpha\, (0 < \alpha \leq 1)$, i = 1, 2 (Eklund and Gahler[9]).

If $\lambda, \mu, \upsilon \geq 0$, then $\lambda(\mu + v) = \lambda\mu + \lambda v$

Further $\lambda \leq \mu \Rightarrow -\mu \leq -\lambda$ and $\lambda + v \leq \mu + v$ for all $v \in R_L$.

Also $\tilde{0} \leq \lambda \leq \mu$ and $\tilde{0} \leq v \Rightarrow \lambda v \leq \mu v$ with respect to $\leq$, inf. $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and sup $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ of a finite numbers of regular fuzzy numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ exist. By $\lambda <^+ \mu$, we mean that $\lambda + \tilde{\delta} \leq \mu$ for some $\delta > 0$ and $(\lambda + \tilde{\delta})(t) = \lambda(t - \delta)$ obviously $u \leq \lambda <^+ \mu \leq v \Rightarrow u <^+ v$. Each regular fuzzy number λ has a unique decomposition $\lambda = \bar{\lambda} + \lambda^+$ in to the 'negative' part $\bar{\lambda}$ and the positive part λ^+ defined by

$$\bar{\lambda}_{\alpha_i} = \min \cdot \left\{ \lambda_{\alpha_i}, 0 \right\}$$

and $$\lambda^{+}_{\alpha_i} = \max \left\{ \lambda_{\alpha_i}, 0) \right\}.$$

This gives us λ^- and λ^+ as regular fuzzy numbers for which $\lambda^- \leq \bar{0} \leq \lambda^+$.

For regular fuzzy numbers, we have the following analogous result which is the consequences of the representation theorem of (Ralescu [6]).

Proposition 2.1 — Let $\lambda_{\alpha_i}, 0 < \alpha \leq 1$ $(i = 1, 2)$ be real numbers.

If (i) $\lambda_{\alpha_1} \leq \lambda \beta_1$ and $\lambda_{\beta_2} \leq \lambda_{\alpha_2}$ for all $\alpha (0 < \alpha \leq 1)$ and for all $\beta (0 < \beta \leq 1)$ with $\alpha < \beta$ and

(ii) $\lim_{\beta \to \alpha} \lambda_{\beta_i} = \lambda_{\alpha_i}$ for all $(0 < \alpha \leq 1)$, $i = 1, 2$.

Then the interval $[\lambda \alpha_1, \lambda \alpha_2]$ represents the level set of a regular L-fuzzy number λ, then the condition (i) and (ii) are satisfied.

Clearly $\lim_{\beta \to \alpha} \lambda_{\beta_i} = \lambda_{\alpha_i}$ means for each $\varepsilon > 0$, there is a $\gamma < \alpha$ such that $| \lambda_{\beta_i} - \lambda_{\alpha_i} | < \varepsilon$ for all β for which $\gamma < \beta \leq \alpha$.

Proposition 2.2 — *Let λ and μ be regular fuzzy numbers. Then*

$$\lambda^- + \mu^- \leq (\lambda + \mu)^-$$

$$(\lambda + \mu)^+ \leq \lambda^+ + \mu^+$$

$$(\lambda \mu)^- = \wedge \left\{ \lambda^- \mu^+, \lambda^+ \mu^+ \right\}$$

$$(\lambda \mu)^+ = \vee \left\{ \lambda^- \mu^-, \lambda^+ \mu^+ \right\}$$

PROOF : Since $\min \left\{ \lambda_{\alpha_i}, 0 \right\} + \min \left\{ \mu_{\alpha_i}, 0 \right\} \leq \min \left\{ \lambda \alpha_i + \mu_{\alpha_i}, 0 \right\}$, the first inequality holds and analogously the second inequality.

Now,

$$[\lambda \mu]^-_\alpha = [\min \left\{ \lambda \alpha_1^- \mu^+_{\alpha_2}, \lambda \alpha_2^+ \mu \alpha_1^- \right\}, \min \left\{ \lambda \alpha_2^- \mu \alpha_1^+ \lambda \alpha_1^+ \mu \alpha_2^- \right\}$$

and $$[\lambda \mu]^+_\alpha = \left[\max \left\{ \lambda \alpha_1^- \mu \alpha_1^-, \lambda \alpha_2^+ \mu \alpha_2^+ \right\}, \max \left\{ \lambda \alpha_2^- \mu \alpha_2^-, \lambda_1^+ \mu \alpha_1^+ \right\} \right];$$

The rest of inequalities thus follow.

A FUZZY METRIC SPACE

All approaches of introducing fuzzy metrics are strongly connected to the environment in which fuzzy metrics are to be applied. We have mainly two different ways of introducing a fuzzy metric. One possibility is to define distances between fuzzy sets (Michael and Eceg[5]) or the Suprium of the Hausdorff distances between corresponding level sets (Diamon and Kloeden[10]) and the other is to fuzzify the distance between crisp points, we choose the latter approach and define fuzzy metrics as fuzzified distance mappings.

Definition 3.1 — By an L-fuzzy metric on a set X we mean a mapping $\rho : X \times X \to R_L$ with the following properties

$$M_1 : p(\lambda, \mu) = \tilde{0} \text{ if and only if } \lambda = \mu$$

$$M_2 = p(\lambda, \mu) = \rho(\mu, \lambda)$$

$$M_3 : \rho(\lambda, \nu) \leq \rho(\lambda, \mu) + \rho(\mu, \nu).$$

Let $\delta(\lambda, \mu)_\alpha = \left| \lambda_{\alpha_1} - \mu_{\alpha_1} \right| + \left| \lambda_{\alpha_2} - \mu_{\alpha_2} \right|$ for all $\lambda, \mu \in R_L$, $0 < \alpha \leq 1$.

Then the mapping $[\cdot] = (\lambda, \mu) \to [(\lambda, \mu)]$ is called a fuzzy metric on R_L we have thus the following proposition.

Proposition 3.2 — *Let* $\lambda, \mu \in R_L$ *and* $[(\lambda, \mu)] = \left[0, \bigwedge_{\beta > \alpha} \delta(\lambda, \mu)(\beta) \right] 0 < \alpha \leq 1$, *are the level sets of a regular fuzzy number* $[(\lambda, \mu)]$, *then* $[\cdot] : (\lambda, \mu) \to [(\lambda, \mu)]$ *is a fuzzy metric on* R_L.

PROOF : Let $\lambda, \mu \in R_L$ and $0 < \alpha \leq 1$

Since

$$\left| \lambda \beta_i - \mu \beta_i \right| \leq | \lambda \alpha_i - \lambda_{1i} | + | \lambda \alpha_i - \mu \alpha_i | = | \mu \alpha_i - \mu_{1i} |,$$

for all $\beta \geq \alpha \, (i = 1, 2)$, $\bigvee_{\beta \geq \alpha} \delta(\lambda, \mu)(\beta)$ is finite.

That $[(\lambda, \mu)]$ is a regular fuzzy number follows from proposition (2.1). We only have to show -

$$\lim_{\beta \to \alpha} \bigvee_{\gamma \geq \beta} \delta(\lambda, \mu)(\gamma) = \bigvee_{\gamma \geq \alpha} \delta(\lambda, \mu)(\gamma)$$

But this is a consequence of

$$\left| \bigvee_{\gamma \geq \beta} \delta(\lambda, \mu)(\gamma) - \bigvee_{\gamma \geq \alpha} \delta(\lambda, \mu)(\gamma) \right| \leq \bigvee_{\substack{\gamma \geq \beta \\ \gamma < \alpha}} | \delta(\lambda, \mu)(\gamma) - \delta(\lambda, u)(\alpha) |$$

and $\lim_{\beta \to \alpha} \delta(\lambda, \mu)(\beta) = \delta(\lambda, \mu)(\alpha)$

Obviously $[\cdot]$ has the properties M_1, M_2, M_3 of fuzzy metric. We will call $[\cdot]$, the cannonical fuzzy metric on R_L to be equipped with $[\cdot]$. Thus proving some technical usability as follows -

Proposition 3.3 — *Let* $\lambda, \mu\ \nu \in R_L$, *then* $[(\lambda + \mu, \lambda + \nu)] = [(\mu, \nu)]$.

PROOF : Obvious

Proposition 3.4 — Let $\lambda, \mu, \lambda', \mu' \in R_L$

Then $[(\lambda + \mu, \lambda' + \mu')] \leq ([\lambda, \lambda') + (\mu, \mu')]$

PROOF : It follows from Proposition (3.3) and

$$[(\lambda + \mu, \lambda' + \mu')] \leq [(\lambda + \mu\ \lambda' + \mu)] + [(\lambda' + \mu, \lambda' + \mu')]$$

Proposition 3.5 — *Let* $\lambda, \mu, \nu \in R_L$

Then $[(\lambda\, \mu, \lambda\, \nu)] \leq [(\bar{0}, \lambda)]\, [(\mu, \nu)]$

PROOF : We have

$$[\lambda\, \mu]^+_\alpha = \left[\ \max\left\{ \lambda\, \bar{\alpha}_1\ \mu\, \bar{\alpha}_1, \lambda\, \alpha_2^+\ \mu\, \alpha_2^+ \right\}, \max\left\{ \lambda\, \bar{\alpha}_2\ \mu\, \bar{\alpha}_2, \lambda\, \alpha_1^+\ \mu\, \alpha_1^+ \right\} \right]$$

and since $|\lambda\, \alpha_i^-|, |\lambda\, \alpha_i^+| \leq [\lambda\, \alpha_i]$ and $\mu\, \alpha_i - \nu\, \alpha_i | = |\mu\, \alpha_i^- + \nu\, \alpha_i^-| + |\mu\, \alpha_i^+ - \nu\, \alpha_i^+|$

hold we get

$$\delta((\lambda\, \mu)^-, (\lambda\, \nu)^-)(\alpha) = |\min(\lambda\, \alpha_1^-\ \mu\, \alpha_2^+, \lambda\, \alpha_2^+\ \mu\, \alpha_1^-)$$

$$- \min(\lambda\, \alpha_1^-\ \nu\, \alpha_2^+, \lambda\, \alpha_2^+\ \nu\, \alpha_1^-)|$$

$$+ |\min(\lambda\, \alpha_2^-\ \mu\, \alpha_1^+, \lambda\, \alpha_1^+\ \mu\, \alpha^-) - \min(\lambda\, \alpha_2^-\ \nu\, \alpha_1^+\ \lambda\, \alpha_1^+\ \nu\, \alpha_2^-)|$$

$$\leq |\lambda\, \alpha_1^-||\mu\, \alpha_2^+ - \nu\, \alpha_2^+| + |\lambda\, \alpha_2^+||\mu\, \alpha_1^- - \nu\, \alpha_1^-|$$

$$+ |\lambda\, \alpha_2^-||\mu\, \alpha_1^+\ \nu\, \alpha_1^+| + |\lambda\, \alpha_1^+||\mu\, \alpha_2^- - \nu\, \alpha_2^-|$$

$$\leq |\lambda\, \alpha_1||\mu\, \alpha_2 - \nu\, \alpha_2| + |\lambda\, \alpha_2||\mu\, \alpha_1 - \nu\, \alpha_1|$$

Analogously,

$\delta((\lambda\, \mu)^+, (\lambda\, \nu)^+)(\alpha) \leq |\lambda\, \alpha_2||\mu\, \alpha_2 - \nu\, \alpha_2| + |\lambda\, \alpha_1||\mu\, \alpha_1 - \nu\, \alpha_1|$ follows

Hence,

$$\delta(\lambda\, \mu, \lambda\, \nu)(\alpha) = \delta((\lambda\, \mu)^-, (\lambda\, \nu)^-)(\alpha) + \delta((\lambda\, \mu)^+, (\lambda\, \nu)^+)(\alpha)$$

$$\leq \delta(\bar{0}, \lambda)(\alpha)\, \delta(\mu, \nu)(\alpha).$$

Therefore, proposition (3.5) holds

Proposition 3.6 — Let $\lambda, \mu, \lambda', \mu' \in R_L$,

$$\text{Then } [(\lambda\,\mu, \lambda'\,\mu')] \leq [(\tilde{0}, \lambda)]\ [(\mu, \mu')] + [(\tilde{0}, \mu)]\ [(\lambda, \lambda')] + ([\lambda, \lambda^1)]\ [(\mu, \mu')]$$

PROOF : It follows from proposition (3.5)

$$[(\lambda\,\mu), (\lambda'\,\mu')] \leq [(\lambda\,\mu, \lambda'\,\mu)] + [(\lambda'\,\mu, \lambda'\,\mu')]$$

and

$$[(\tilde{0}, \lambda')] \leq [(\tilde{0}, \lambda)] + [(\lambda, \lambda')]$$

Remark 3.7 : The *n*-dimensional L-fuzzy real space R_L^n will be developed in another paper.

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