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SOME NEW GENERALIZATIONS OF HILBERT'S INTEGRAL INEQUALITIES

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In this paper, some generalizations of Hilbert's integral inequalities are showed by introducing two real functions $\phi(x)$ and $\psi(x)$.

Key Words : Hilbert's integral inequality

INTRODUCTION

The following inequalities are well-known as Hilbert's integral inequality :

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy \le \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right)^{1/2} \quad \ldots (1)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx \right)^2 dy \le \pi^2 \int_0^\infty f^2(x)\,dx \quad \ldots (2)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y+1}\,dx\,dy \le \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right)^{1/2} \quad \ldots (3)$$

where π is the best value (Cf. [1, Chap 9]).

In recent years, Hu[2], Gao[3] and Kuang[4] gave some distinct improvements of (1).

Yang Bicheng[5] gave interesting generalization of (1) by introducing parameters $\lambda \in (0, 1]$,

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}\,dx\,dy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^\infty t^{1-\lambda} f^2(t)\,dt \int_0^\infty t^{1-\lambda} g^2(t)\,dt\right)^{1/2} \quad \ldots (4)$$

where $B(p, q)$ is the β function.

Yang Bicheng[6] made some generalizations of (1) and (4) by introducing three parameters, A, B and λ.

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(Ax+By)^\lambda}\,dx\,dy < \frac{1}{(AB)^{\lambda/2}} B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$$

$$\left(\int_0^\infty x^{1-\lambda} f^2(x)\,dx \int_0^\infty y^{1-\lambda} g^2(y)\,dy\right)^{1/2} \quad \ldots (5)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{(Ax+By)^\lambda}\,dx\right)^2 dy < \frac{1}{(AB)^\lambda}\left(B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right)^2 \int_0^\infty x^{1-\lambda} f^2(x)\,dx \quad \ldots (6)$$

Where $A, B, \lambda > 0$.

Gao Mingzhe[7] gave the following improvement of (3) by introducing

$$\alpha(x) = 2arctg\frac{1}{\sqrt{2x+1}},$$

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y+1}\,dx\,dy \le$$

$$\left(\int_0^\infty (\pi - \alpha(x)) f^2(x)\,dx\right)^{1/2}\left(\int_0^\infty (\pi - \alpha(y))\,g^2(y)\,dy\right)^{1/2} \quad \ldots (7)$$

Yang Bicheng[8] gave the following generalization of (1)and (3) by introducing parameters

$\alpha, \beta, \lambda\ (\alpha \ge -\beta, \lambda \in (1/r, 1]\ (r = p, q))$.

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(x+y+2\beta)^\lambda}\,dx\,dy \le$$

$$\left\{\int_\alpha^\infty \left(\tilde{K}_\lambda(p) - \ln 2\left(\frac{\alpha+\beta}{t+\beta}\right)^{1/p}\right)(t+\beta)^{1-\lambda} f^p(t)\,dt\right\}^{1/p}$$

$$\times \left\{ \int_{\alpha}^{\infty} \left(\tilde{K}_{\lambda}(q) - \ln 2 \left(\frac{\alpha+\beta}{t+\beta} \right)^{1/q} \right) (t+\beta)^{1-\lambda} g^{q}(t)\, dt \right\}^{1/q} \quad \ldots (8)$$

where $$\tilde{K}_{\lambda}(r) = \int_{0}^{\infty} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u} \right)^{1-1/r} \quad (r = p, q).$$

Yang Bicheng[9] gave the following more advanced generalization.

$$\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)\, g(y)}{(x+y+2\beta)^{\lambda}}\, dx\, dy \leq$$

$$\left\{ \int_{\alpha}^{\infty} \left(\tilde{K}_{\lambda}(p) - \theta_{\lambda}(q) \left(\frac{\alpha+\beta}{t+\beta} \right)^{1+\frac{\lambda-2}{q}} \right) (t+\beta)^{1-\lambda} f^{p}(t)\, dt \right\}^{1/p} \times$$

$$\left\{ \int_{\alpha}^{\infty} \left(\tilde{K}_{\lambda}(q) - \theta_{\lambda}(p) \left(\frac{\alpha+\beta}{t+\beta} \right)^{1+\frac{\lambda-2}{p}} \right) (t+\beta)^{1-\lambda} g^{q}(t)\, dt \right\}^{1/q} \quad \ldots (9)$$

where $\alpha \geq -\beta, \lambda > 2 - \min\{p, q\}$ and $K_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right)$ is the best value.

In this paper, we show some new generalizations on above inequalities by introducing two real functions $\phi(x)$ and $\psi(x)$.

First we introduce some Lemmas :

Lemma 1[9] — Let $a < 1, \lambda > 0$, define $h(y)$ as

$$h(y) = y^{-1+a} \int_{0}^{y} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u} \right)^{a} + du, y \in (0, 1]$$

Then $h(y) > h(1)$ $(0 < y < 1)$.

Lemma 2 — Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(\alpha) \geq -\beta, \phi'(x) > 0, \psi(\alpha) \geq -\beta, \psi'(x) > 0, A, B > 0, \lambda > 2 - \min\{p, q\}$, define

$$\omega(\phi, \psi, A, B, \lambda, \alpha, \beta, q, x) \text{ and } \omega(\psi, \phi, A, B, \lambda, \alpha, \beta, p, y)$$

$$\text{as } \omega(\phi, \psi, A, B, \lambda, \alpha, \beta, q, x) = \int_{\alpha}^{\infty} \frac{1}{(A\phi(x) + B\psi(y) + (A+B)\beta)^{\lambda}}$$

$$\left(\frac{A(\phi(x)+\beta)}{B(\psi(y)+\beta)} \right)^{\frac{2-\lambda}{q}} dy, x > \alpha \cdot$$

$$\omega(\psi, \phi, A, B, \lambda, \alpha, \beta, p, y) = \int_{\alpha}^{\infty} \frac{1}{(A\phi(x) + B\psi(y) + (A+B)\beta)^{\lambda}}$$

$$\left(\frac{B(\psi(y)+\beta)}{A(\phi(x)+\beta)}\right)^{\frac{2-\lambda}{p}} dx,\ y > \alpha$$

then

(i) when $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$, we have

$$\omega(\phi, \psi, A, B, \lambda, \alpha, \beta, q, x) = \frac{(A\phi(x) + A\beta)^{1-\lambda}}{B\psi'(y)} B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) \quad \ldots (10)$$

$$\omega(\psi\phi, A, B, \lambda, \alpha, \beta, p, y) = \frac{(B\psi(y) + B\beta)^{1-\lambda}}{A\phi'(x)} B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) \quad \ldots (11)$$

(ii) when $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$, we have

$$\omega(\phi, \psi, A, B, \lambda, \alpha, \beta, q, x) < \frac{(A\phi(x) + A\beta)^{1-\lambda}}{B\psi'(y)}$$

$$\left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) - \theta(\lambda, q)\left(\frac{B(\psi(\alpha)+\beta)}{A(\phi(x)+\beta)}\right)^{\frac{q+\lambda-2}{q}}\right) \quad \ldots (12)$$

$$\omega(\psi, \phi, A, B, \lambda, \alpha, \beta, p, y) < \frac{(B\psi(y) + B\beta)^{1-\lambda}}{A\phi'(x)}$$

$$\left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) - \theta(\lambda, p)\left(\frac{A(\phi(\alpha)+\beta)}{B(\psi(y)+\beta)}\right)^{\frac{p+\lambda-2}{p}}\right) \quad \ldots (13)$$

where $\theta(\lambda, r) = \int_0^1 \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}$, $r = p, q$.

PROOF : Putting $u = \frac{B(\psi(y)+\beta)}{A(\phi(x)+\beta)}$, we have

$$\omega(\phi, \psi, A, B, \lambda, \alpha, \beta, q, x) = \frac{(A\phi(x) + A\beta)^{1-\lambda}}{B\psi'(y)}$$

$$\int_{\frac{B(\psi(\alpha)+\beta)}{A(\phi(x)+\beta)}}^{\infty} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{\frac{2-\lambda}{q}} du$$

(i) when $\phi(\alpha) = -\beta$, (10) is valid.

(ii) when $\phi(\alpha) > -\beta$, by Lemma 1 we have

$$\omega(\phi, \psi, A, B, \lambda, \alpha, \beta, q, x)$$

$$= \frac{(A\phi(x)+A\beta)^{1-\lambda}}{B\psi'(y)}\left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right)-\left(\frac{B(\psi(\alpha)+\beta)}{A(\phi(x)+\beta)}\right)^{\frac{q+\lambda-2}{q}}\right.$$

$$\left.\int_0^{\frac{B(\psi(\alpha+\beta)}{A(\phi(x)+\beta)}} \frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{q}} du\right)$$

$$< \frac{(A\phi(x)+A\beta)^{1-\lambda}}{B\psi'(y)}\left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right)-\theta(\lambda, q)\left(\frac{B(\psi(\alpha)+\beta)}{A(\phi(x)+\beta)}\right)^{\frac{q+\lambda-2}{q}}\right)$$

Then (12) is valid. Similaraly (11) and (13) can be proved.

By Lemma 2 we have the following results.

Corollary 3 — Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, define $\omega(\phi, \psi, \alpha, \beta, q, x)$ and $\omega(\psi, \phi, \alpha, \beta, p, y)$ as

$$\omega(\phi, \psi, \alpha, \beta, q, x) = \int_{\alpha}^{\infty} \frac{1}{\phi(x)+\psi(y)+2\beta}\left(\frac{\phi(x)+\beta}{\psi(y)+\beta}\right)^{\frac{1}{q}} dy, x > \alpha$$

$$\omega(\psi, \phi, \alpha, \beta, p, y) = \int_{\alpha}^{\infty} \frac{1}{\phi(x)+\psi(y)+2\beta}\left(\frac{\psi(y)+\beta}{\phi(x)+\beta}\right)^{\frac{1}{p}} dx, y > \alpha$$

then

(i) when $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$, we have

$$\omega(\phi, \psi, \alpha, \beta, q, x) = \frac{B\left(\frac{1}{p}, \frac{1}{q}\right)}{\psi'(y)}. \quad \ldots (14)$$

$$\omega(\psi, \phi, \alpha, \beta, p, y) = \frac{B\left(\frac{1}{p}, \frac{1}{q}\right)}{\phi'(x)} \quad \ldots (15)$$

(ii) when $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$, we have

$$\omega(\phi, \psi, \alpha, \beta, q, x) = \frac{1}{\psi'(y)}\left(B\left(\frac{1}{p}, \frac{1}{q}\right)-\theta(q)\left(\frac{\psi(\alpha)+\beta}{\phi(x)+\beta}\right)^{\frac{1}{p}}\right) \quad \ldots (16)$$

$$\omega(\psi, \phi, \alpha, \beta, p, y) = \frac{1}{\psi'(x)}\left(B\left(\frac{1}{p}, \frac{1}{q}\right) - \theta(p)\left(\frac{\phi(\alpha)+\beta}{\psi(y)+\beta}\right)^{\frac{1}{q}} \right) \quad \ldots (17)$$

where $\theta(r) = \int_0^1 \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{r}}, r = p, q.$

Corollary 4 — Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, $\lambda > 2 - \min\{p, q\}$, define $\omega(\phi, \psi, \lambda, \alpha, \beta, q, x)$ and $\omega(\psi, \phi, \lambda, \alpha, \beta, p, y)$ as

$$\omega(\phi, \psi, \lambda, \alpha, \beta, q, x) = \int_\alpha^\infty \frac{1}{(\phi(x)+\psi(y)+2\beta)^\lambda}\left(\frac{\phi(x)+\beta}{\psi(y)+\beta}\right)^{\frac{2-\lambda}{q}} dy, x > \alpha$$

$$\omega(\psi, \phi, \lambda, \alpha, \beta, p, y) = \int_\alpha^\infty \frac{1}{(\phi(x)+\psi(y)+2\beta)^\lambda}\left(\frac{\psi(y)+\beta}{\phi(x)+\beta}\right)^{\frac{2-\lambda}{p}} dx, y > \alpha$$

then

(i) when $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$, we have

$$\omega(\phi, \psi, \lambda, \alpha, \beta, q, x) = \frac{(\phi(x)+\beta)^{1-\lambda}}{\psi'(y)} B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) \quad \ldots (18)$$

$$\omega(\psi, \phi, \lambda, \alpha, \beta, p, y) = \frac{(\psi(y)+\beta)^{1-\lambda}}{\phi'(x)} B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) \quad \ldots (19)$$

(ii) when $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$, we have

$$\omega(\phi, \psi, \lambda, \alpha, \beta, q, x) < \frac{(\phi(x)+\beta)^{1-\lambda}}{\psi'(y)}\left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) - \theta(\lambda, q)\left(\frac{\psi(\alpha)+\beta}{\phi(x)+\beta}\right)^{\frac{q+\lambda-2}{q}} \right) \quad \ldots (20)$$

$$\omega(\psi, \phi, \lambda, \alpha, \beta, p, y) < \frac{(\psi(y)+\beta)^{1-\lambda}}{\phi'(X)}\left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) \right.$$

$$- \theta(\lambda, p)\left(\frac{\phi(\alpha)+\beta}{\psi(y)+\beta}\right)^{\frac{p+\lambda-2}{p}}\Bigg) \qquad \ldots (21)$$

where $\theta(\lambda, r) = \int_0^1 \frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}, r = p, q.$

Corollary 5 — Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, A, $B > 0$, define $\omega(\phi, \psi, A, B, \alpha, \beta, q, x)$ and $\omega(\psi, \phi, A, B, \alpha, \beta, p, y)$ as

$$\omega(\phi, \psi, A, B, \alpha, \beta, q, x) = \int_{\alpha}^{\infty} \frac{1}{(A\,\phi(x) + B\,\psi(y) + (A+B)\,\beta}$$

$$\left(\frac{A\,(\phi(x)+\beta)}{B\,(\psi(y)+\beta)}\right)^{\frac{1}{q}} dy,\ x > \alpha$$

$$\omega(\psi, \phi, A, B, \alpha, \beta, p, y) = \int_{\alpha}^{\infty} \frac{1}{A\,\phi(x) + B\,\psi(y) + (A+B)\,\beta}$$

$$\left(\frac{B\,(\psi(y)+\beta)}{A\,(\phi(x)+\beta)}\right)^{\frac{1}{p}} dx,\ y > \alpha$$

then

(i) when $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$ we have

$$\omega(\phi, \psi, A, B, \alpha, \beta, q, x) = \frac{1}{B\,\psi'(y)} B\left(\frac{1}{p}, \frac{1}{q}\right) \qquad \ldots (22)$$

$$\omega(\psi, \phi, A, B, \alpha, \beta, p, y) = \frac{1}{A\,\phi'(x)} B\left(\frac{1}{p}, \frac{1}{q}\right) \qquad \ldots (23)$$

(ii) when $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$, we have

$$\omega(\phi, \psi, A, B, \alpha, \beta, q, x) < \frac{1}{B\,\psi'(y)}\left(B\left(\frac{1}{p}, \frac{1}{q}\right) - \theta(q)\left(\frac{B\,(\psi(\alpha)+\beta)}{A\,(\phi(x)+\beta)}\right)^{\frac{1}{p}}\right) \qquad \ldots (24)$$

$$\omega(\psi, \phi, A, B, \alpha, \beta, p, y) < \frac{1}{A\,\phi'(x)}\left(B\left(\frac{1}{p}, \frac{1}{q}\right) - \theta(p)\left(\frac{A(\phi(\alpha)+\beta)}{B(\psi(y)+\beta)}\right)^{\frac{1}{q}}\right) \quad \text{... (25)}$$

where $\theta(r) = \int_0^1 \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{r}}$, $r = p, q$.

MAIN RESULTS

Now we introduce main results

Theorem 1 — *Let $p > 1$, $1/p + 1/q = 1$, f and g be real functions, $\phi(x)$ and $\psi(x)$ be differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, such that*

$$0 < \int_\alpha^\infty f^p(t)\,dt < \infty \text{ and } 0 < \int_\alpha^\infty g^q(t)\,dt < \infty.$$

Then

(i) when $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{\phi(x)+\psi(y)+2\beta}\,dx\,dy \leq \frac{B\left(\frac{1}{p}, \frac{1}{q}\right)}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\left(\int_\alpha^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_\alpha^\infty g^q(y)\,dy\right)^{\frac{1}{q}} \quad \text{... (26)}$$

(ii) When $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$,we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{\phi(x)+\psi(y)+2\beta}\,dx\,dy < \frac{1}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\left(\int_\alpha^\infty \left[B\left(\frac{1}{p}, \frac{1}{q}\right) - \theta(q)\left(\frac{\psi(\alpha)+\beta}{\phi(x)+\beta}\right)^{\frac{1}{p}}\right] f^p(x)\,dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\alpha}^{\infty}\left[B\left(\frac{1}{p},\frac{1}{q}\right)-\theta(p)\left(\frac{\phi(\alpha)+\beta}{\psi(y)+\beta}\right)^{\frac{1}{q}}\right]g^{q}(y)\,dy\right)^{\frac{1}{q}} \qquad \ldots (27)$$

PROOF : By Hölder's inequality and Corollary 3, we have

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{\phi(x)+\psi(y)+2\beta}\,dx\,dy$$

$$=\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)}{(\phi(x)+\psi(y)+2\beta)^{\frac{1}{p}}}\left(\frac{\phi(x)+\beta}{\psi(y)+\beta}\right)^{\frac{1}{pq}}$$

$$\frac{g(y)}{(\phi(x)+\psi(y)+2\beta)^{\frac{1}{q}}}\left(\frac{\psi(y)+\beta}{\phi(x)+\beta}\right)^{\frac{1}{pq}}dx\,dy$$

$$\leq\left(\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f^{p}(x)}{\phi(x)+\psi(y)+2\beta}\left(\frac{\phi(x)+\beta}{\psi(y)+\beta}\right)^{\frac{1}{q}}dx\,dy\right)^{\frac{1}{p}}$$

$$\left(\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{g^{q}(y)}{\phi(x)+\psi(y)+2\beta}\left(\frac{\psi(y)+\beta}{\phi(x)+\beta}\right)^{\frac{1}{p}}dx\,dy\right)^{\frac{1}{q}}$$

$$=\left(\int_{\alpha}^{\infty}\omega(\phi,\psi,\alpha,\beta,q,x)f^{p}(x)\,dx\right)^{\frac{1}{p}}\left(\int_{\alpha}^{\infty}\omega(\psi,\phi\,\alpha,\beta,p,y)\,g^{q}(y)\,dy\right)^{\frac{1}{q}}$$

When $\phi(\alpha)=-\beta$, $\psi(\alpha)=-\beta$, we have

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{\phi(x)+\psi(y)+2\beta}\,dx\,dy\leq\frac{B\left(\frac{1}{p},\frac{1}{q}\right)}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\left(\int_{\alpha}^{\infty}f^{p}(x)\,dx\right)^{\frac{1}{p}}\left(\int_{\alpha}^{\infty}g^{q}(y)\,dy\right)^{\frac{1}{q}}$$

Hence (26) is valid.

When $\phi(\alpha)>-\beta$, $\psi(\alpha)>-\beta$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{\phi(x)+\psi(y)+2\beta}\,dx\,dy < \frac{1}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\left(\int_\alpha^\infty \left[B\left(\frac{1}{p},\frac{1}{q}\right)-\theta(q)\left(\frac{\psi(\alpha)+\beta}{\phi(x)+\beta}\right)^{\frac{1}{p}}\right]f^p(x)\,dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_\alpha^\infty \left[B\left(\frac{1}{p},\frac{1}{q}\right)-\theta(p)\left(\frac{\phi(\alpha)+\beta}{\psi(y)+\beta}\right)^{\frac{1}{q}}\right]g^q(y)\,dy\right)^{\frac{1}{q}}$$

Hence (27) is valid. This completes the proof of the theorem.

Theorem 2 — *Let $p > 1$, $1/p + 1/q = 1$, f and g be real functions, $\phi(x)$ and $\psi(x)$ be differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, $\lambda > 2$ - min {p, q}, such that*

$$0 < \int_\alpha^\infty (\phi(t)+\beta)^{1-\lambda} f^p(t)\,dt < \infty \text{ and } 0 < \int_\alpha^\infty (\psi(t)+\beta)^{1-\lambda} g^q(t)\,dt < \infty.$$

Then

(i) when $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\,dxdy \leq \frac{B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\left(\int_\alpha^\infty (\phi(x)+\beta)^{1-\lambda} f^p(x)\,dx\right)^{\frac{1}{p}} \times \left(\int_\alpha^\infty (\psi(y)+\beta)^{1-\lambda} g^q(y)\,dy\right)^{\frac{1}{q}} \quad \ldots (28)$$

(ii) when $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\,dxdy < \frac{1}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\times\left[\int_\alpha^\infty \left[B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)-\theta(\lambda,q)\left(\frac{\psi(\alpha)+\beta}{\phi(x)+\beta}\right)^{\frac{q+\lambda-2}{q}}\right](\phi(x)+\beta)^{1-\lambda} f^p(x)\,dx\right]^{\frac{1}{p}}$$

$$\times\left(\int_\alpha^\infty \left[B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)-\theta(\lambda,p)\left(\frac{\phi(\alpha)+\beta}{\psi(y)+\beta}\right)^{\frac{p+\lambda-2}{p}}\right](\psi(y)+\beta)^{1-\lambda} g^q(y)\,dy\right)^{\frac{1}{q}} \quad \ldots (29)$$

PROOF : By Hölder inequality and Corollary 4, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\,dx\,dy$$

$$= \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)}{(\phi(x)+\psi(y)+2\beta)^{\frac{\lambda}{p}}}\left(\frac{\phi(x)+\beta}{\psi(y)+\beta}\right)^{\frac{2-\lambda}{pq}}$$

$$\frac{g(y)}{(\phi(x)+\psi(y)+2\beta)^{\frac{\lambda}{q}}}\left(\frac{\psi(y)+\beta}{\phi(x)+\beta}\right)^{\frac{2-\lambda}{pq}}\,dxdy$$

$$\le \left(\int_\alpha^\infty \int_\alpha^\infty \frac{f^p(x)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\left(\frac{\phi(x)+\beta}{\psi(y)+\beta}\right)^{\frac{2-\lambda}{q}}\,dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_\alpha^\infty \int_\alpha^\infty \frac{g^q(y)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\left(\frac{\psi(y)+\beta}{\phi(x)+\beta}\right)^{\frac{2-\lambda}{p}}\,dx\,dy\right)^{\frac{1}{q}}$$

$$= \left(\int_\alpha^\infty \omega(\phi,\psi,\lambda,\alpha,\beta,q,x)\,f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_\alpha^\infty \omega(\psi,\phi,\lambda,\alpha,\beta,p,y)\,g^q(y)\,dy\right)^{\frac{1}{q}}$$

When $\phi(\alpha)=-\beta$, $\psi(\alpha)=-\beta$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\,dx\,dy \le$$

$$\frac{B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}\left(\int_\alpha^\infty (\phi(x)+\beta)^{1-\lambda} f^p(x)\,dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_\alpha^\infty (\psi(y)+\beta)^{1-\lambda} g^q(y)\,dy\right)^{\frac{1}{q}}$$

Hence (28) is valid.

When $\phi(\alpha)>-\beta$, $\psi(\alpha)>-\beta$, we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(\phi(x)+\psi(y)+2\beta)^\lambda}\,dx\,dy < \frac{1}{(\phi'(x))^{\frac{1}{q}}(\psi'(y))^{\frac{1}{p}}}$$

$$\times\left(\int_\alpha^\infty\left[B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)-\theta(\lambda,q)\left(\frac{\psi(\alpha)+\beta}{\phi(x)+\beta}\right)^{\frac{q+\lambda-2}{q}}\right](\phi(x)+\beta)^{1-\lambda}f^p(x)\,dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_\alpha^\infty\left[B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)-\theta(\lambda,p)\left(\frac{\phi(\alpha)+\beta}{\psi(y)+\beta}\right)^{\frac{p+\lambda-2}{p}}\right](\psi(y)+\beta)^{1-\lambda}g^q(y)\,dy\right)^{\frac{1}{q}}$$

Hence (29) is valid. The theorem is proved.

In a similar way, the following theorems can be shown.

Theorem 3 — *Let $p > 1$, $1/p + 1/q = 1$, f and g be real functions, $\phi(x)$ and $\psi(x)$ be differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, $A, B > 0$, such that*

$$0 < \int_\alpha^\infty f^p(t)\,dt < \infty \text{ and } 0 < \int_\alpha^\infty g^q(t)\,dt < \infty.$$

Then

$$\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{A\phi(x)+B\psi(y)+(A+B)\beta}\,dxdy \leq \frac{B\left(\frac{1}{p},\frac{1}{q}\right)}{(A\phi'(x))^{\frac{1}{q}}(B\psi'(y))^{\frac{1}{p}}}\times$$

$$\left(\int_\alpha^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_\alpha^\infty g^q(y)\,dy\right)^{\frac{1}{q}} \qquad \ldots (30)$$

When $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$ we have

$$\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{A\phi(x)+B\psi(y)+(A+B)\beta}\,dxdy \leq \frac{1}{(A\phi'(x))^{\frac{1}{q}}(B\psi'(y))^{\frac{1}{p}}}$$

$$\times\left(\int_\alpha^\infty\left(B\left(\frac{1}{p},\frac{1}{q}\right)-\theta(q)\left(\frac{B(\psi(\alpha)+\beta)}{A(\phi(X)+\beta)}\right)^{\frac{1}{p}}\right)f^p(x)\,dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_\alpha^\infty\left(B\left(\frac{1}{p},\frac{1}{q}\right)-\theta(p)\left(\frac{B(\phi(\alpha)+\beta)}{B(\psi(y)+\beta)}\right)^{\frac{1}{q}}\right)g^q(y)\,dy\right)^{\frac{1}{q}} \qquad \ldots (31)$$

Theorem 4 — *Let $p > 1$, $1/p + 1/q = 1$, f and g be real functions, $\phi(x)$ and $\psi(x)$ be differentiable functions, and $\phi(\alpha) \geq -\beta$, $\phi'(x) > 0$, $\psi(\alpha) \geq -\beta$, $\psi'(x) > 0$, $A, B > 0$, $\lambda > 2 - \min\{p, q\}$,*

such that $0 < \int_{\alpha}^{\infty} \phi^{1-\lambda}(t) f^{p}(t)\, dt < \infty$ *and* $0 < \int_{\alpha}^{\infty} \psi^{1-\lambda}(t) g^{q}(t)\, dt < \infty$. *Then*

When $\phi(\alpha) = -\beta$, $\psi(\alpha) = -\beta$, *we have*

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{f(x)g(y)}{(A\phi(x) + B\psi(y) + (A+B)\beta)}\, dxdy \le \left(\frac{B^{1-\lambda}}{A\phi'(x)}\right)^{\frac{1}{q}} \left(\frac{A^{1-\lambda}}{B\psi'(y)}\right)^{\frac{1}{p}}$$

$$\times B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) \left(\int_{\alpha}^{\infty} (\phi(x)+\beta)^{1-\lambda} f^{p}(x)\, dx\right)^{\frac{1}{p}}$$

$$\left(\int_{\alpha}^{\infty} (\psi(y)+\beta)^{1-\lambda} g^{q}(y) dy\right)^{\frac{1}{q}} \qquad \ldots (32)$$

When $\phi(\alpha) > -\beta$, $\psi(\alpha) > -\beta$, *we have*

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{f(x)g(y)}{(A\phi(x) + B\psi(y) + (A+B)\beta)^{\lambda}}\, dxdy < \left(\frac{B^{1-\lambda}}{A\phi'(x)}\right)^{\frac{1}{q}} \left(\frac{A^{1-\lambda}}{B\psi'(y)}\right)^{\frac{1}{p}}$$

$$\times \left(\int_{\alpha}^{\infty} \left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) - \theta(\lambda, q)\left(\frac{B(\psi(\alpha)+\beta)}{A(\phi(x)+\beta)}\right)^{\frac{q+\lambda-2}{q}}\right)\right.$$

$$\left. (\phi(x)+\beta)^{1-\lambda} f^{p}(x)\, dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_{\alpha}^{\infty} \left(B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) - \theta(\lambda, p)\left(\frac{A(\phi(\alpha)+\beta)}{B(\psi(y)+\beta)}\right)^{\frac{p+\lambda-2}{p}}\right)\right.$$

$$\left. (\psi(y)+\beta)^{1-\lambda} g^{q}(y)\, dy\right)^{\frac{1}{q}} \qquad \ldots (33)$$

Remark 1 : For $\phi(x) = \psi(x) = x$, $p = q = 2$, $\beta = \alpha = 0$, $\lambda = 1$, inequality (26) changes to (1) and (28) changes to (1), hence inequalities (26) and (28) are generalizations of (1).

Remark 2 : For $\phi(x) = \psi(x) = x$, $p = q = 2$, $\beta = \alpha = 0$, inequality (28) changes to (4), hence inequality (28) is generalization of (4).

Remark 3 : For $\phi(x) = \psi(x) = x, p = q = 2, \beta = \alpha = 0$, inequalities (32) changes to (5), hence inequaly (32) is generalization of (5).

Remark 4 : For $\phi(x) = \psi(x) = x, p = q = 2, \beta = \frac{1}{2}, \alpha = 0, \lambda = 1$, inequalities (27) and (29) change to (7), hence inequalities (27) and (29) are generalizations of (7).

Remark 5 : For $\phi(x) = \psi(x) = x, A = B = 1$, inequalities (29) and (33) change to (9) hence inequalities (29) and (33) are generalizations of (9).

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